Daniel Čech, BI48

Matematická analýza - zápočtové příklady

```
> restart;
> with(plots):
> with(plottools):
> with(student):
> with(linalg):
> with(DEtools):
> with(inttrans):
>
```

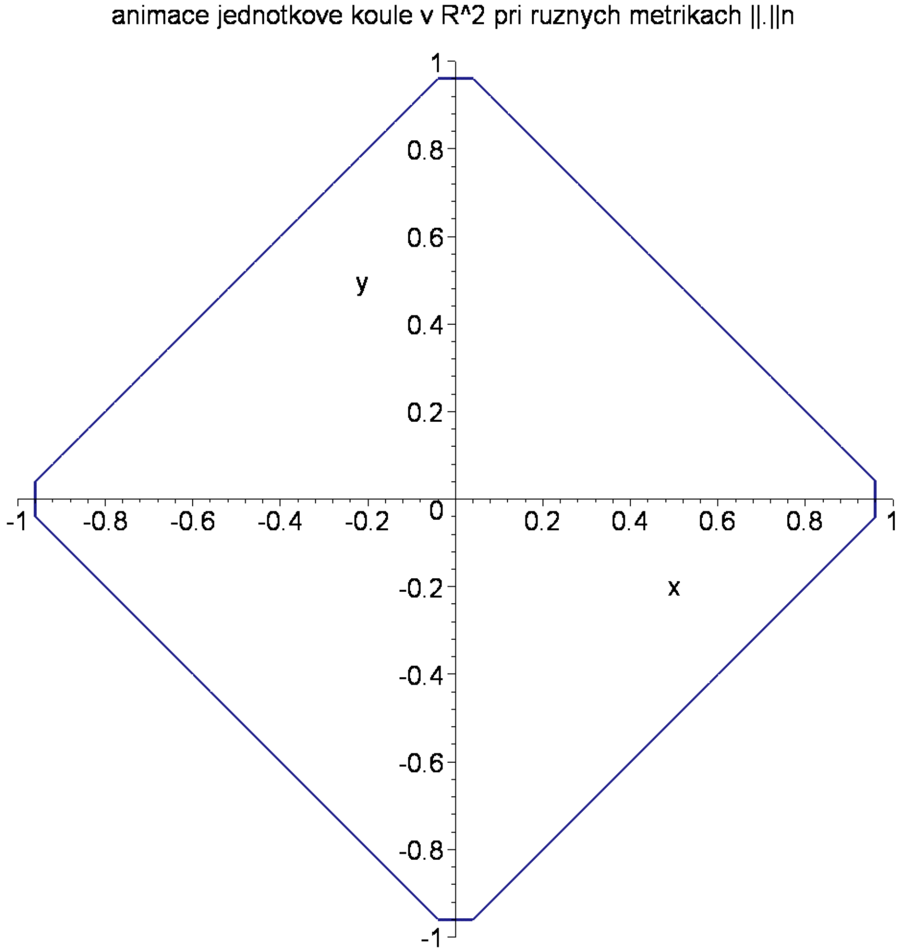
Metrické prostory

```
>
```

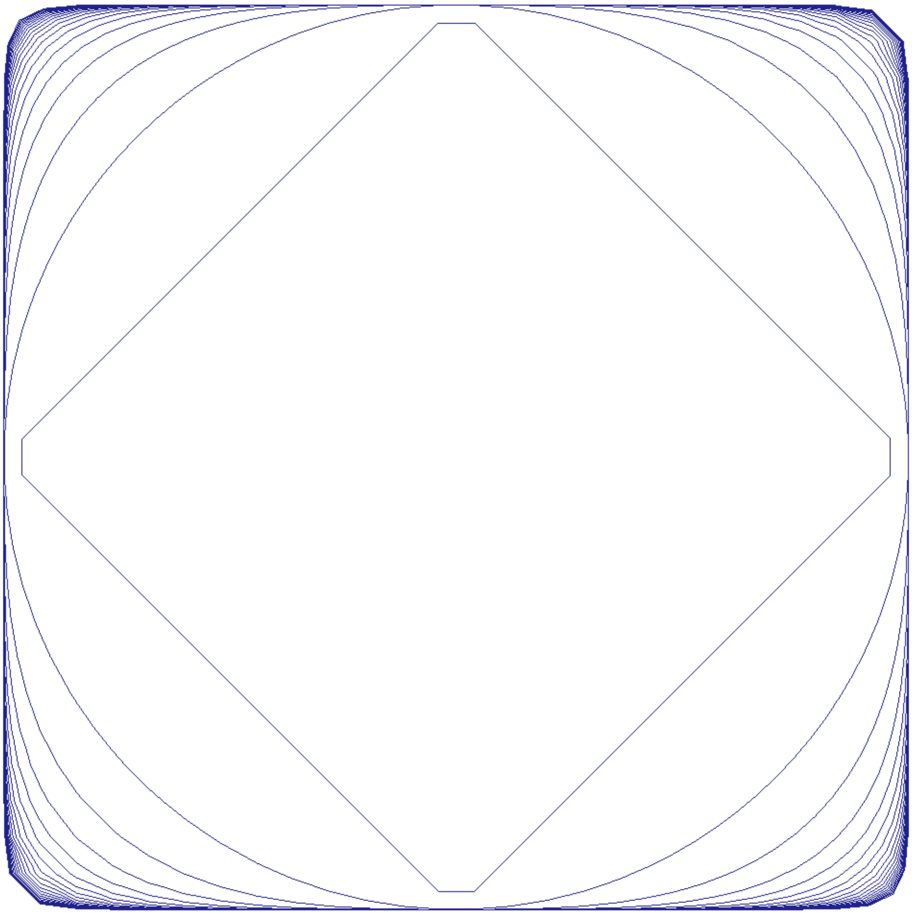
1) v *R* mají všechny body stejnou vzdálenost od počátku při použití všech norem, platí totiž $\|x\|n=(|x|^n)^{\left(\frac{1}{n}\right)}=|x|$.

2) R^2 a R^3

```
> posloupnost1:=seq(implicitplot((abs(x)^m+abs(y)^m)^(1/m)=1,x=
-1..1,y=-1..1,colour=navy),m=1..20):
> display(posloupnost1,insequence=true,title=`animace
jednotkove koule v R^2 pri ruznych metrikach
||.||n`,thickness=3,scaling=constrained);
```

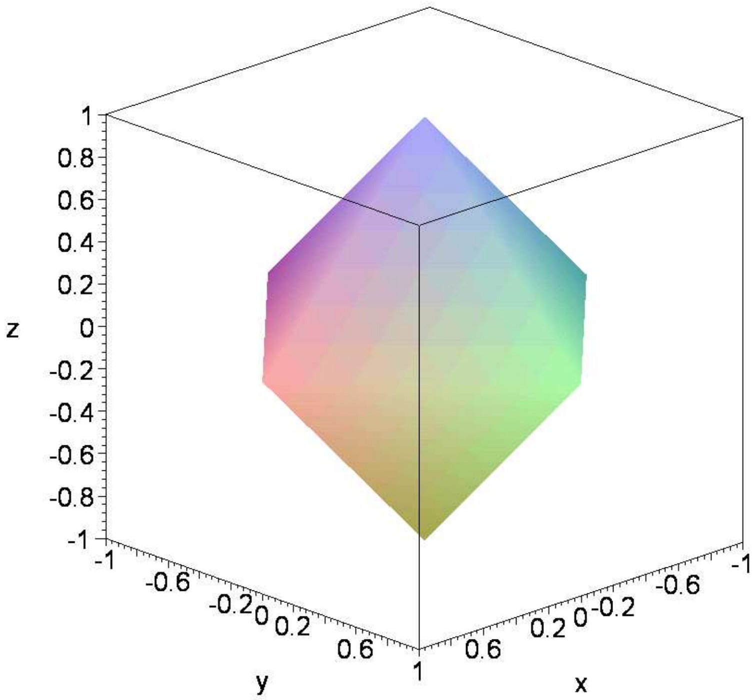



```
> display(posloupnost1,insequence=false,scaling=constrained,axe
  s=none);
```



```
> posloupnost2:=seq(implicitplot3d((abs(x)^m+abs(y)^m+abs(z)^m)
  ^(1/m)=1,x=-1..1,y=-1..1,z=-1..1,grid=[15,15,15],style=patchn
  ogrid),m=1..10):
> display(posloupnost2,insequence=true,scaling=constrained,axes
  =boxed,title=`animace jednotkove koule v R^3 pri ruznych
  metrikach ||.||n`,orientation=[46,70]);
```

animace jednotkove koule v R^3 pri ruznych metrikach ||.||n




```
> 
> m:=5;
```

$$m := 5$$

```
> rho:=(xvect,yvect)->(sum(abs(xvect[i]-yvect[i])^g,
  i=1..m))^(1/g);
```

$$\rho := (xvect, yvect) \to \left(\sum_{i=1}^{m} |xvect_i - yvect_i|^g\right)^{\left(\frac{1}{g}\right)}$$

```
> sigma:=(xvect,yvect)->(sum(abs(xvect[i]-yvect[i])^h,
  i=1..m))^(1/h);
```

$$\sigma := (xvect, yvect) \to \left(\sum_{i=1}^{m} |xvect_i - yvect_i|^h\right)^{\left(\frac{1}{h}\right)}$$

```
> vx:=[seq(x[i],i=1..m)];
```

$$vx := [x_1, x_2, x_3, x_4, x_5]$$

```
> vy:=[seq(y[i],i=1..m)];
```

$$vy := [y_1, y_2, y_3, y_4, y_5]$$

```
> rho(vx,vy); sigma(vx,vy);
```

$$\left(|x_1-y_1|^g+|x_2-y_2|^g+|x_3-y_3|^g+|x_4-y_4|^g+|x_5-y_5|^g\right)^{\left(\frac{1}{g}\right)}$$

$$\left(|x_1-y_1|^h+|x_2-y_2|^h+|x_3-y_3|^h+|x_4-y_4|^h+|x_5-y_5|^h\right)^{\left(\frac{1}{h}\right)}$$

```
> solve(df*rho(x,y)=sigma(x,y),df);
```

$$\frac{\left(|x_1-y_1|^h+|x_2-y_2|^h+|x_3-y_3|^h+|x_4-y_4|^h+|x_5-y_5|^h\right)^{\left(\frac{1}{h}\right)}}{\left(|x_1-y_1|^g+|x_2-y_2|^g+|x_3-y_3|^g+|x_4-y_4|^g+|x_5-y_5|^g\right)^{\left(\frac{1}{g}\right)}}$$

```
>
>
> rx:=randvector(m); ry:=randvector(m); g:=3: h:=2: #za g a h
  lze zadat jakekoliv prirozene cislo
```

$rx := [-10, 62, -82, 80, -44]$

$ry := [71, -17, -75, -10, -7]$

```
> evalf(rho(rx,ry)); evalf(sigma(rx,ry));
```

121.7447858

149.3987952

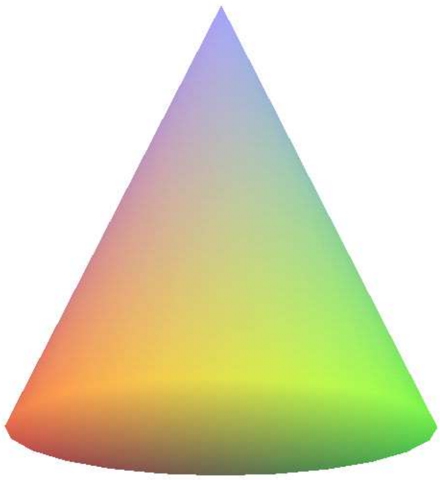
Křivkový integrál 1. druhu

```
>
```

Povrch podstavy je jednoduchý : $S_p = \pi\, r^2 = \pi$

```
> plast:=plot3d([r*sin(Phi),r*cos(Phi),2-2*r],r=0..1,Phi=0..2*P
  i):
> podstava:=plot3d([r*sin(Phi),r*cos(Phi),0],r=0..1,Phi=0..2*Pi
  ):
> display(plast,podstava,scaling=constrained,style=patchnogrid,
  title=`kuzel:`,orientation=[44,103]);
```

kuzel:



>

$$\int 1\, dS = \int_{D}^{?} \left| \left(\frac{\partial}{\partial r} S\right) x \left(\frac{\partial}{\partial \phi} S\right) \right| d\phi$$

```
> dSdr:=vector([diff(r*cos(phi),r),diff(r*sin(phi),r),diff(2-2*r,r)]);
```

$$dSdr := [\cos(\phi), \sin(\phi), -2]$$

```
> dSdphi:=vector([diff(r*cos(phi),phi),diff(r*sin(phi),phi),diff(2-2*r,phi)]);
```

$$dSdphi := [-r \sin(\phi), r \cos(\phi), 0]$$

```
> prod:=crossprod(dSdr,dSdphi);
```

$$prod := [2\, r \cos(\phi), 2\, r \sin(\phi), \cos(\phi)^2\, r + \sin(\phi)^2\, r]$$

```
> absize:=sqrt(prod[1]^2+prod[2]^2+prod[3]^3);
```

$$absize := \sqrt{4\, r^2 \cos(\phi)^2 + 4\, r^2 \sin(\phi)^2 + (\cos(\phi)^2\, r + \sin(\phi)^2\, r)^3}$$

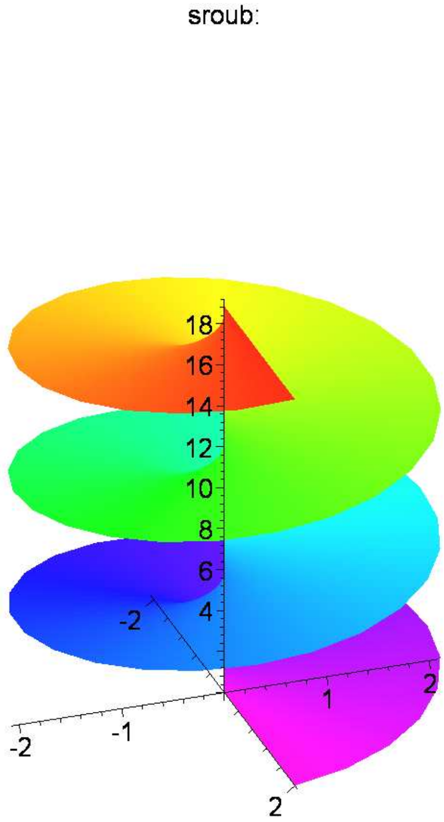
```
> int(int(absize, r=0..1), phi=0..2*Pi);
```

$$\frac{256}{15}\pi - \frac{20}{3}\sqrt{5}\,\pi$$

Což je hledaný povrch pláště kuželu, kužel má proto povrch $-\frac{20\sqrt{5}\,\pi}{3} + \frac{256\,\pi}{15} + \pi$

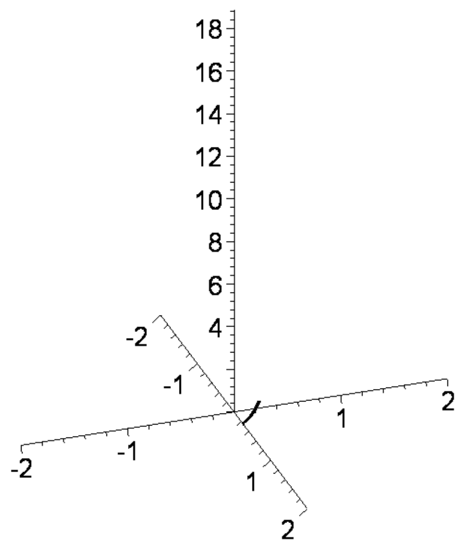
>

```
> sroub:=plot3d([r*cos(phi),r*sin(phi),phi],r=0..2,phi=0..6*Pi,
  numpoints=5000,style=patchnogrid, shading=zhue,axes=normal,
  orientation=[-19,63],title=`sroub:`):
> krivka:=seq(seq(spacecurve([n/4*cos(t),n/4*sin(t),t],t=(m-1)*
  Pi/4..m*Pi/4,colour=black,thickness=4),n=1..8),m=1..24):
> vlna:=display(krivka,insequence=true):
> display(sroub);
```




```
> display(vlna,insequence=true,title=`animovany postup
  integrace:`,axes=normal, orientation=[-19,63]);
```

animovany postup integrace:




```
> dSdr:=vector([diff(r*cos(phi),r),diff(r*sin(phi),r),diff(h*r,r)]);
```

$$dSdr := [\cos(\phi), \sin(\phi), 2]$$

```
> dSdphi:=vector([diff(r*cos(phi),phi),diff(r*sin(phi),phi),diff(h*r,phi)]);
```

$$dSdphi := [-r\sin(\phi), r\cos(\phi), 0]$$

```
> prod:=crossprod(dSdr,dSdphi);
```

$$prod := [-2\,r\cos(\phi), -2\,r\sin(\phi), \cos(\phi)^2\,r + \sin(\phi)^2\,r]$$

```
> absize:=sqrt(prod[1]^2+prod[2]^2+prod[3]^2);
```

$$absize := \sqrt{4\,r^2\cos(\phi)^2 + 4\,r^2\sin(\phi)^2 + (\cos(\phi)^2\,r + \sin(\phi)^2\,r)^2}$$

$$\int 1\,dS = \int_D^{?} \left| \left(\frac{\partial}{\partial r} S\right) x \left(\frac{\partial}{\partial \phi} S\right) \right| d\phi$$

podle Fubiniovy věty potom platí:

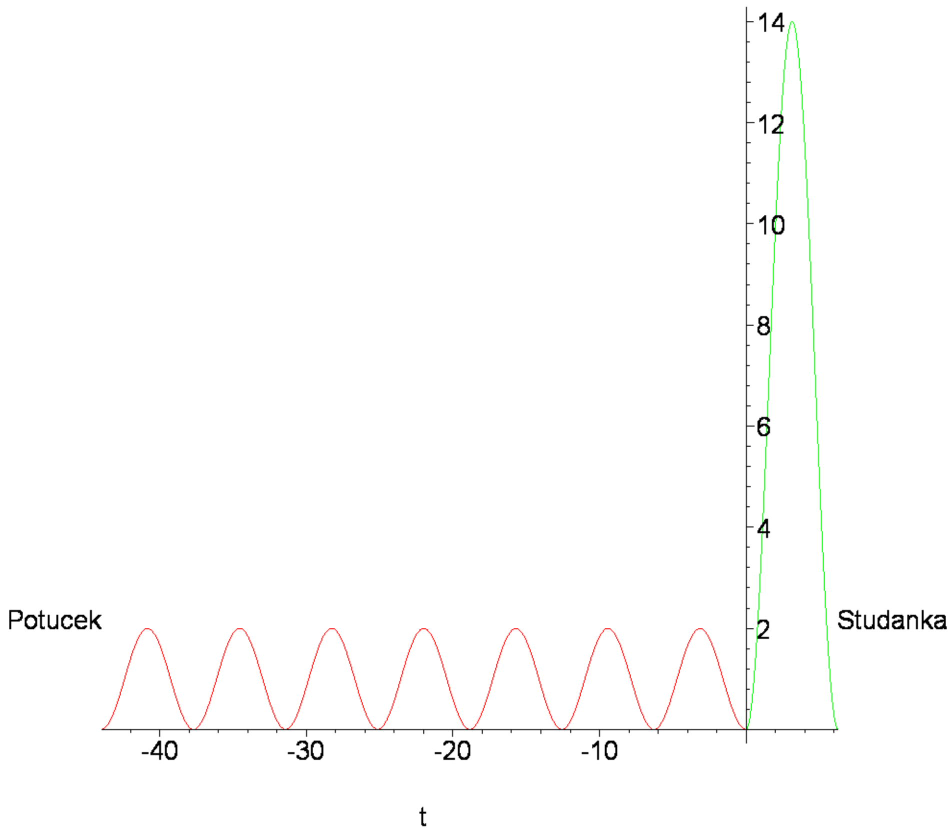
$$\int_0^{6\pi}\int_0^2 \sqrt{h^2\,r^2\cos(\phi)^2 + h^2\,r^2\sin(\phi)^2 + (\cos(\phi)^2\,r + \sin(\phi)^2\,r)^2}\;dr\,d\phi$$

```
> int(int(absize, r=0..a), phi=0..6*Pi);
```

$$3\,\pi\,a^2\sqrt{5}\,\mathrm{csgn}(a)$$

Což je hledaná plocha závislá na parametrech *h* a a

```
>
> velky:=plot(7-7*cos(t),t=0..2*Pi, colour=green):
> male:=plot(1-cos(t),t=-14*Pi..0, colour=red):
> krajina:=display(velky,male,textplot([2*Pi,2,`Studanka`],align={ABOVE,RIGHT}),textplot([-14*Pi,2,`Potucek`],align={ABOVE,LEFT})):
> display(krajina);
```




```
> diff(7-7*cos(t),t);
```

$$7\sin(t)$$

```
> diff(1-cos(t),t);
```

$$\sin(t)$$

```
>
```

doba prochazky= $\int_{t_{min}}^{t_{max}} \left(1-\left(\frac{\partial}{\partial t} f\right)\right)\sqrt{1-\left(\frac{\partial}{\partial ft} f\right)}\, dt$

```
> int(abs(sin(t))*sqrt(1+sin(t)),t=-14*Pi..0);
```

$$\frac{112}{3}-\frac{28\sqrt{2}}{3}$$

```
> evalf(%);
```

$$24.13400675$$

```
> int(abs(7*sin(t))*sqrt(1+7*sin(t)),t=0..2*Pi);
```

$$\frac{56}{3}-\frac{32}{3} I\sqrt{2}\sqrt{7}\,\mathrm{EllipticF}\left(\frac{\sqrt{2}\sqrt{3}\sqrt{7}}{6}, \frac{\sqrt{3}\sqrt{7}}{7}\right) + \frac{8}{3} I\sqrt{2}\sqrt{7}\,\mathrm{EllipticE}\left(\frac{\sqrt{2}\sqrt{3}\sqrt{7}}{6}, \frac{\sqrt{3}\sqrt{7}}{7}\right) + \frac{16}{3} I\sqrt{7}\sqrt{2}\,\mathrm{EllipticF}\left(\frac{\sqrt{3}\sqrt{7}}{3}, \frac{\sqrt{3}\sqrt{7}}{7}\right) - \frac{4}{3} I\sqrt{7}\sqrt{2}\,\mathrm{EllipticE}\left(\frac{\sqrt{3}\sqrt{7}}{3}, \frac{\sqrt{3}\sqrt{7}}{7}\right) + \frac{32}{3} I\sqrt{7}\sqrt{2}\,\mathrm{EllipticK}\left(\frac{\sqrt{3}\sqrt{7}}{7}\right) - \frac{8}{3} I\sqrt{7}\sqrt{2}\,\mathrm{EllipticE}\left(\frac{\sqrt{3}\sqrt{7}}{7}\right)$$

```
>
```

```
> dPhixdt:=diff(t,t);
```

$$dPhixdt := 1$$

```
> dPhiydt:=diff(7-7*cos(t),t);
```

$$dPhiydt := 7\sin(t)$$

```
> ivalue:=innerprod([abs(7*sin(t)),1],[dPhixdt,dPhiydt]);
```

$$ivalue := 7\left|\sin(t)\right| + 7\sin(t)$$

```
> int(ivalue,t=0..2*Pi);
```

$$28$$

Podle toho, jak to vychází, je jedno, jestli půjde slepička k potůčku nebo ke studánce, protože obě cesty jsou stejně dlouhé.

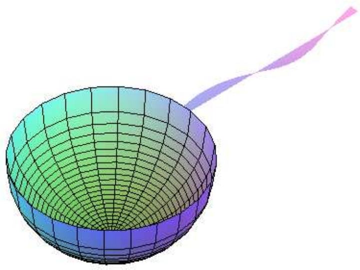
Křivkový integrál 2. druhu

```
>
```

$\int F\, dS = \int_{D}^{?} \mathrm{F}(\mathrm{S}(u, v)) \left(\frac{\partial}{\partial r} S\right) x \left(\frac{\partial}{\partial \phi} S\right) d\phi$

```
> cednik:=plot3d([r*cos(phi),r*sin(phi),-sqrt(1-r^2)],r=0..1,ph
```

```
i=0..2*Pi,style=patch,title=`cednik:`):
> rukojet:=plot3d([phi/3+1,r*cos(phi),r*sin(phi)],r=0..0.1,phi=
0..2*Pi,style=patchnogrid):
> display(cednik,rukojet,scaling=constrained,orientation=[-131,
47]);
```




```
> dSdu:=vector([diff(u,u),diff(v,u),diff(-sqrt(1-u^2-v^2),u)]);
```

$$dSdu := \left[1, 0, \frac{u}{\sqrt{1-u^2-v^2}} \right]$$

```
> dSdv:=vector([diff(u,v),diff(v,v),diff(-sqrt(1-u^2-v^2),v)]);
```

$$dSdv := \left[0, 1, \frac{v}{\sqrt{1-u^2-v^2}} \right]$$

```
> prod:=crossprod(dSdu,dSdv);
```

$$prod := \left[-\frac{u}{\sqrt{1-u^2-v^2}}, -\frac{v}{\sqrt{1-u^2-v^2}}, 1 \right]$$

```
> ivalue:=innerprod([u,v,-sqrt(1-u^2-v^2)],prod);
```

$$ivalue := -\frac{1}{\sqrt{1-u^2-v^2}}$$

```
>
> int(int(ivalue*r, r=0..1), phi=0..2*Pi);
```

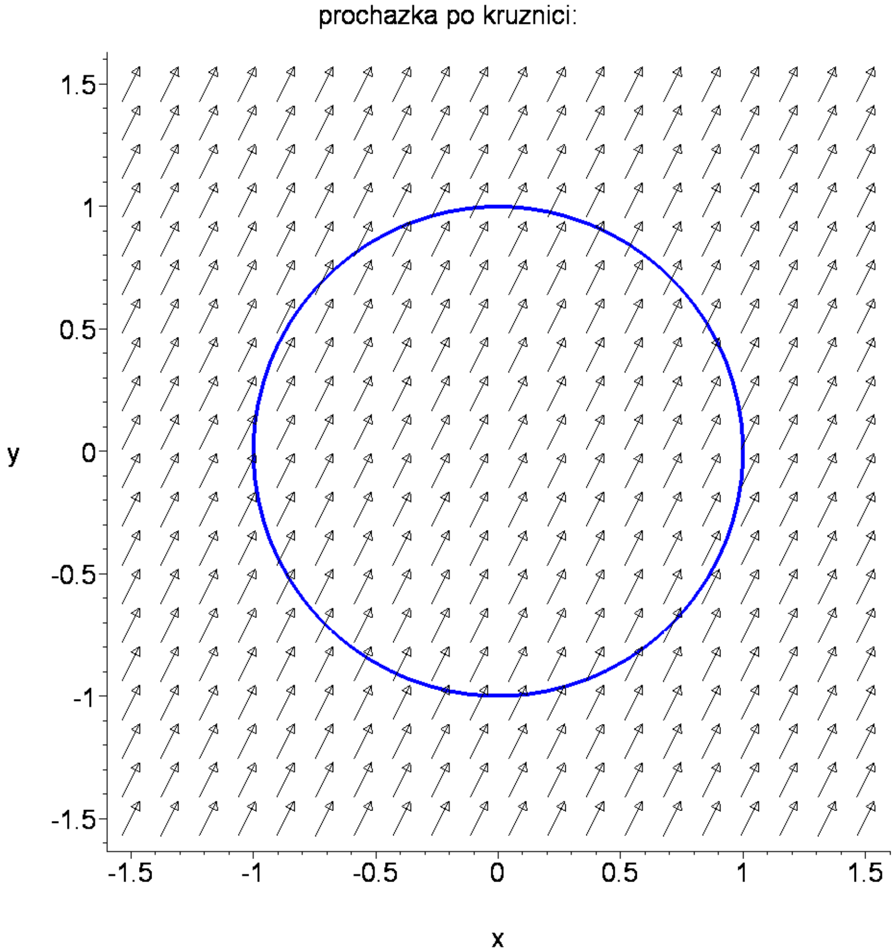
$$-\frac{\pi}{\sqrt{1-u^2-v^2}}$$

To je hledaný průtok.

>

$$\int F\,d\Phi = \int_0^{2\pi} \mathrm{F}(\Phi(t))\left[\frac{\partial}{\partial t} Phi\_1, \frac{\partial}{\partial t} Phi\_2\right] dt$$

```
> pole:=fieldplot( [1,2],x=-1.5..1.5,y=-1.5..1.5,arrows=slim):
> kruh:=implicitplot(x^2+y^2=1,x=-1..1,y=-1..1,colour=blue,thic
  kness=4):
> display(pole,kruh,scaling=constrained,axes=frame,title=`proch
  azka po kruznici:`);
```




```
> dPhidt:=diff([cos(t),sin(t)],t);
```

$$dPhidt := [-\sin(t), \cos(t)]$$

```
> ivalue:=innerprod([a,b],dPhidt);
```

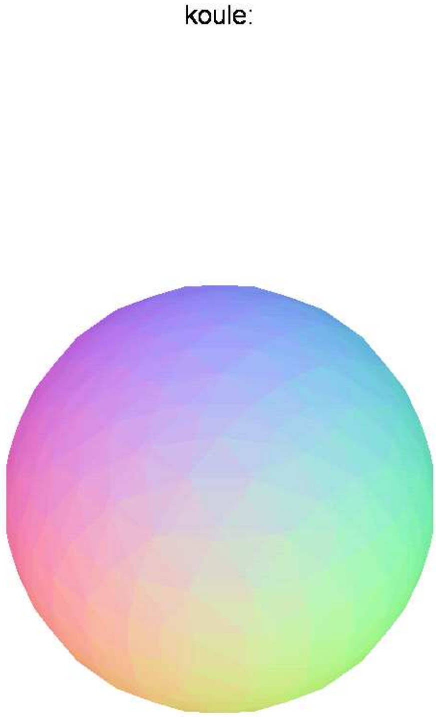
$$ivalue := -a\sin(t) + b\cos(t)$$

```
> int(ivalue, t=0..2*Pi);
```

$$0$$

práce vykonaná při procházce po kruhu za konstantního větru je nulová

```
>
> implicitplot3d(x^2+y^2+z^2=1,x=-1..1,y=-1..1,z=-1..1,scaling=
  constrained,style=patchnogrid,title=`koule:`);
```




```
>
> divergence:=diff(x,x)+diff(y,y)+diff(z,z);
```

$$divergence := 3$$

$$\int_{M_1}^{?} 3\, dxdydz = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} 3\, dx\, dy\, dz$$

M_2={(r, θ, ϕ)εR^3: 0<=r<=1, -π/2<=θ<=π/2, 0<=ϕ<=2π }

```
> jacob:=jacobian([r*cos(theta)*cos(phi),r*cos(theta)*sin(phi),
  r*sin(theta)],[r,theta,phi]);
```

$$jacob := \begin{bmatrix} \cos(\theta)\cos(\phi) & -r\sin(\theta)\cos(\phi) & -r\cos(\theta)\sin(\phi) \\ \cos(\theta)\sin(\phi) & -r\sin(\theta)\sin(\phi) & r\cos(\theta)\cos(\phi) \\ \sin(\theta) & r\cos(\theta) & 0 \end{bmatrix}$$

```
> determinant:=det(jacob):
>
> abs(int(int(int(3*determinant,r = 0 .. 1),theta = -Pi/2 ..
  Pi/2),phi = 0 .. 2*Pi));
```

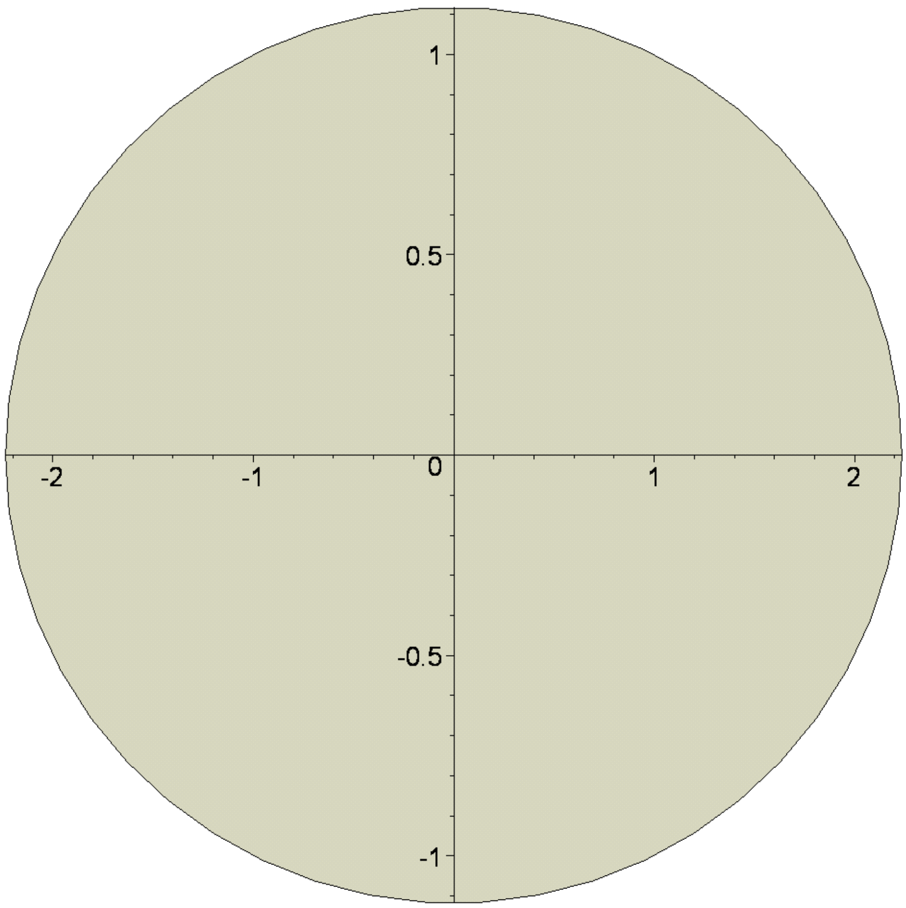
$$4\pi$$

4π je hledaný povrch jednotkové koule.

```
>

> vnitrek:=ellipse([0,0], sqrt(5), sqrt(5/4), filled=true,
  color=wheat):
> sek:=display(seq(plot([sqrt(5)*cos(t),sqrt(5/4)*sin(t),t=0..n
  /5*Pi],colour=blue,thickness=4),n=0..10),insequence=true):
> display(sek,vnitrek,title=`animovana prochazka po kraji
  elipsy:`);
```

animovana prochazka po kraji elipsy:




```
>
```

křivka Φ(t) je hladká na celém intervalu <0,2π>

```
> Phi:=t->[sqrt(5)*cos(t),sqrt(5/4)*sin(t),0];
```

$$\Phi := t \to \left[\sqrt{5}\cos(t), \sqrt{\frac{5}{4}}\sin(t), 0\right]$$

```
> tau:=t->[-sqrt(5)*sin(t),sqrt(5/4)*cos(t),0];
```

$$\tau := t \to \left[-\sqrt{5}\sin(t), \sqrt{\frac{5}{4}}\cos(t), 0\right]$$

```
> crossprod([0,0,1],tau(t));
```

$$\left[-\frac{1}{2}\sqrt{5}\cos(t), -\sqrt{5}\sin(t), 0\right]$$

```
>
> dx:=f->diff(f,x): dy:=f->diff(f,y): dz:=f->diff(f,z):
> rotace:=[dy(y)-dz(x),dz(z)-dx(y),dx(x)-dy(z)];
```

$$rotace := [1, 1, 1]$$

$$\int \mathrm{rot}(F)\, dS = \int_D^{?} \mathrm{rot}(\mathrm{F}(\mathrm{S}(u,v))) \left(\frac{\partial}{\partial u} S\right) x \left(\frac{\partial}{\partial v} S\right) dudv$$

```
> normala:=crossprod([1,0,0],[0,1,0]);
```

$$normala := [0, 0, 1]$$

```
> ivalue:=innerprod([1,1,1],normala);
```

$$ivalue := 1$$

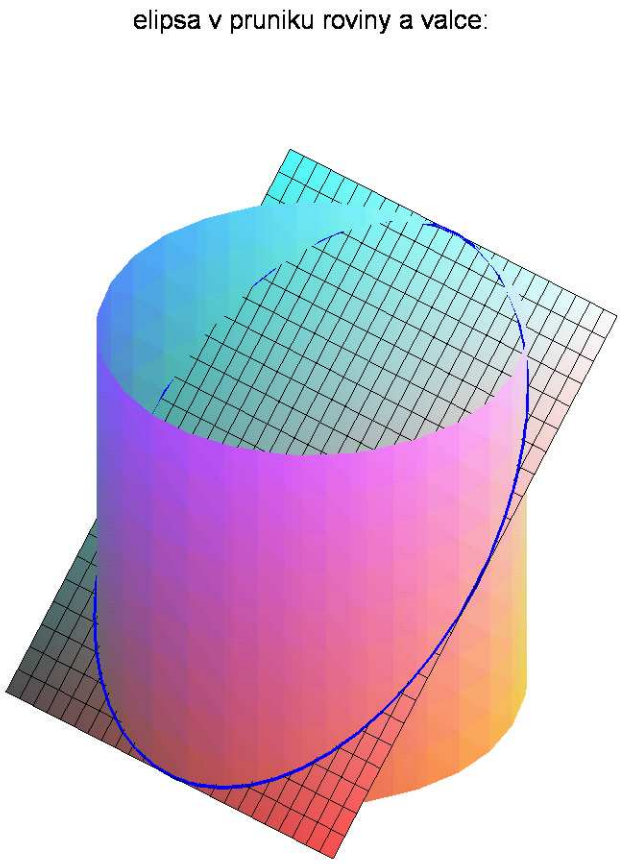
$$\int_{D}^{?} 1\, dudv = \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{5-v^2}}^{\sqrt{5-v^2}} 1\, du\, dv$$

```
> int(int(ivalue,u = -sqrt(5-v^2) .. sqrt(5-v^2)),v = -sqrt(5)
  .. sqrt(5));
```

$$5\,\pi$$

Výsledek je $\int F\, d\Phi = 5\pi$

```
>
> valec:=implicitplot3d(x^2+y^2=1,x=-1..1,y=-1..1,z=2..4,style=
  patchnogrid):
> rovina:=plot3d([x,y,y+3],x=-1..1,y=-1..1):
> elipsa:=spacecurve([cos(t),sin(t),sin(t)+3],t=0..2*Pi,thickne
  ss=4,color=blue):
> display(valec,rovina,elipsa,orientation=[-49,54],title=`elips
  a v pruniku roviny a valce:`);
```




```
> 
> rotace:=[dy(-2*y)-dz(5*x),dz(3*z)-dx(-2*y),dx(5*x)-dy(3*z)];
```

$$rotace := [-2, 3, 5]$$

```
> S:=(u,v)->[u,v,v+3]:
> normala:=crossprod(diff(S(u,v),u),diff(S(u,v),v));
```

$$normala := [0, -1, 1]$$

```
> ivalue:=innerprod(rotace,normala);
```

$$ivalue := 2$$

$$\int \mathrm{rot}(F)\, dS = \int_{D}^{?} 2\, dudv = \int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} 2\, du\, dv$$

```
> int(int(2,u = -sqrt(1-v^2) .. sqrt(1-v^2)),v = -1 .. 1);
```

$$2\pi$$

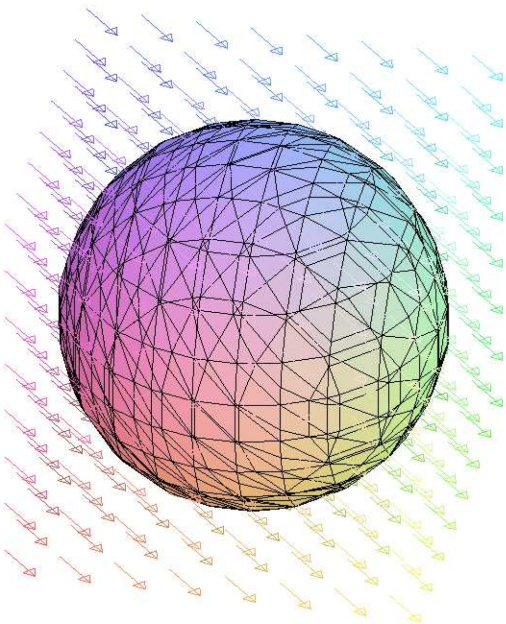
2π je hodnota integrálu $\int F\, d\Phi$

```
> 
```

```
>
> pole:=fieldplot3d([1,1,0],x=-1..1,y=-1..1,z=-1..1,arrows=SLIM
  ):
  koule:=implicitplot3d(x^2+y^2+z^2=1,x=-1..1,y=-1..1,z=-1..1):

  display(pole,koule,orientation=[12,56],title=`koule ve
  vektorovem poli:`,scaling=constrained);
```

koule ve vektorovem poli:



$$\int F\, dS = \int_D^{?} \mathrm{F}(\mathrm{S}(\phi, \theta)) \left(\frac{\partial}{\partial \phi} S \right) x \left(\frac{\partial}{\partial \theta} S \right) dudv$$

```
> S:=(phi,theta)->[cos(theta)*cos(phi), cos(theta)*sin(phi),
  sin(theta)];
```

$$S := (\phi, \theta) \to [\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), \sin(\theta)]$$

```
> dSdphi:=diff(S(phi,theta),phi);
```

$$dSdphi := [-\cos(\theta)\sin(\phi), \cos(\theta)\cos(\phi), 0]$$

```
> dSdtheta:=diff(S(phi,theta),theta);
```

$$dSdtheta := [-\sin(\theta)\cos(\phi), -\sin(\theta)\sin(\phi), \cos(\theta)]$$

```
> normala:=crossprod(dSdphi,dSdtheta);
```

$$normala := [\cos(\theta)^2\cos(\phi), \cos(\theta)^2\sin(\phi), \cos(\theta)\sin(\phi)^2\sin(\theta) + \cos(\theta)\cos(\phi)^2\sin(\theta)]$$

```
> ivalue:=innerprod([1,1,0],normala);
```

$$ivalue := \cos(\theta)^2 \cos(\phi) + \cos(\theta)^2 \sin(\phi)$$

$$\int F\, dS = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)^2 \cos(\phi) + \cos(\theta)^2 \sin(\phi)\, d\theta\, d\phi$$

```
> int(int(ivalue,theta = -Pi/2 .. Pi/2),phi = 0 .. 2*Pi);
```

$$0$$

Integrál je roven 0.

```
>
>
```

$$\int F\, dS = \int_M^{?} \mathrm{div}(F)\, dxdydz$$

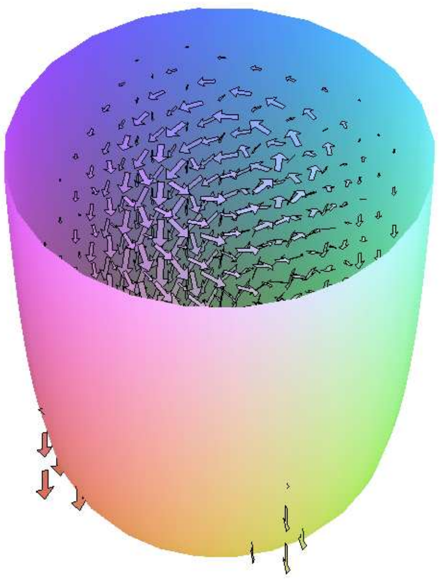
```
> divergence:=dx(1)+dy(1)+dz(0);
```

$$divergence := 0$$

Protože je divergence 0 je integrál roven 0.

```
>
> umyvadlo:=plot3d([(3*v)^(1/8)*cos(phi),(3*v)^(1/8)*sin(phi),v
  ],v=0.00001..1,phi=0..2*Pi,style=patchnogrid,title=`umyvadlo:
  `):
> vir:=fieldplot3d([((1-sqrt(x^2+y^2))^2)*(-y),((1-sqrt(x^2+y^2
  ))^2)*x,z/6-1/6],x=-0.7..0.7,y=-0.7..0.7,z=0.1..1,arrows=THIC
  K):
> display(umyvadlo,vir);
```

umyvadlo:



>

$$\int F\,dS = \int_{D}^{?} \mathrm{F}(\mathrm{S}(u,v)) \left(\frac{\partial}{\partial u} S\right) x \left(\frac{\partial}{\partial v} S\right) dudv$$

```
> S:=(u,v)->[u,v,0];
```

$$S := (u, v) \to [u, v, 0]$$

```
> dSdu:=diff(S(u,v),u);
```

$$dSdu := [1, 0, 0]$$

```
> dSdv:=diff(S(u,v),v);
```

$$dSdv := [0, 1, 0]$$

```
> normala:=crossprod(dSdu,dSdv);
```

$$normala := [0, 0, 1]$$

```
> F:=(x,y,z)->[((1-sqrt(x^2+y^2))^2*(-y), (1-sqrt(x^2+y^2))^2*x
, z/6-1/6 )];
```

$$F := (x, y, z) \to \left[-(1-\sqrt{y^2+x^2})^2 y, (1-\sqrt{y^2+x^2})^2 x, \frac{1}{6} z - \frac{1}{6}\right]$$

```
> ivalue:=-1/6:
>
>
> int(int(-1/6,u = -sqrt(1-v^2) .. sqrt(1-v^2)),v = -1 .. 1);
```

$$-\frac{\pi}{6}$$

Okamžitý průtok vody kanálkem je -1/6π

Laplaceova transformace

```
> with(inttrans):
```

```
> obraz:=laplace((D@@3)(y)(t)+(D@@2)(y)(t)=6*t+exp(-t),t,s);
```

$$obraz := s^3\,\mathrm{laplace}(y(t), t, s) - (D^{(2)})(y)(0) - s\,D(y)(0) - s^2\,y(0) + s^2\,\mathrm{laplace}(y(t), t, s) - D(y)(0) - s\,y(0) = \frac{6}{s^2} + \frac{1}{1+s}$$

```
> y0:=0: dy0:=1: ddy0:=1:
```

```
> s*(s*(s*laplace(y(t),t,s)-y(0))-D(y)(0))-`@@`(D,2)(y)(0)+s*(s
*laplace(y(t),t,s)-y(0))-D(y)(0) = 6*1/(s^2)+1/(1+s);
```

$$s\,(s\,(s\,\mathrm{laplace}(y(t), t, s) - y(0)) - D(y)(0)) - (D^{(2)})(y)(0) + s\,(s\,\mathrm{laplace}(y(t), t, s) - y(0)) - D(y)(0) = \frac{6}{s^2} + \frac{1}{1+s}$$

```
> solve(s*(s*(s*laplace(y(t),t,s)-y(0))-D(y)(0))-`@@`(D,2)(y)(0
)+s*(s*laplace(y(t),t,s)-y(0))-D(y)(0) =
6*1/(s^2)+1/(1+s),laplace(y(t),t,s));
```

$$(6 + 6\,s + s^2 + 2\,s^4\,y(0) + s^5\,y(0) + 2\,s^3\,D(y)(0) + s^4\,D(y)(0) + D(y)(0)\,s^2 + (D^{(2)})(y)(0)\,s^2 + (D^{(2)})(y)(0)\,s^3 + s^3\,y(0)) \Big/ (s^4\,(1+s)^2)$$

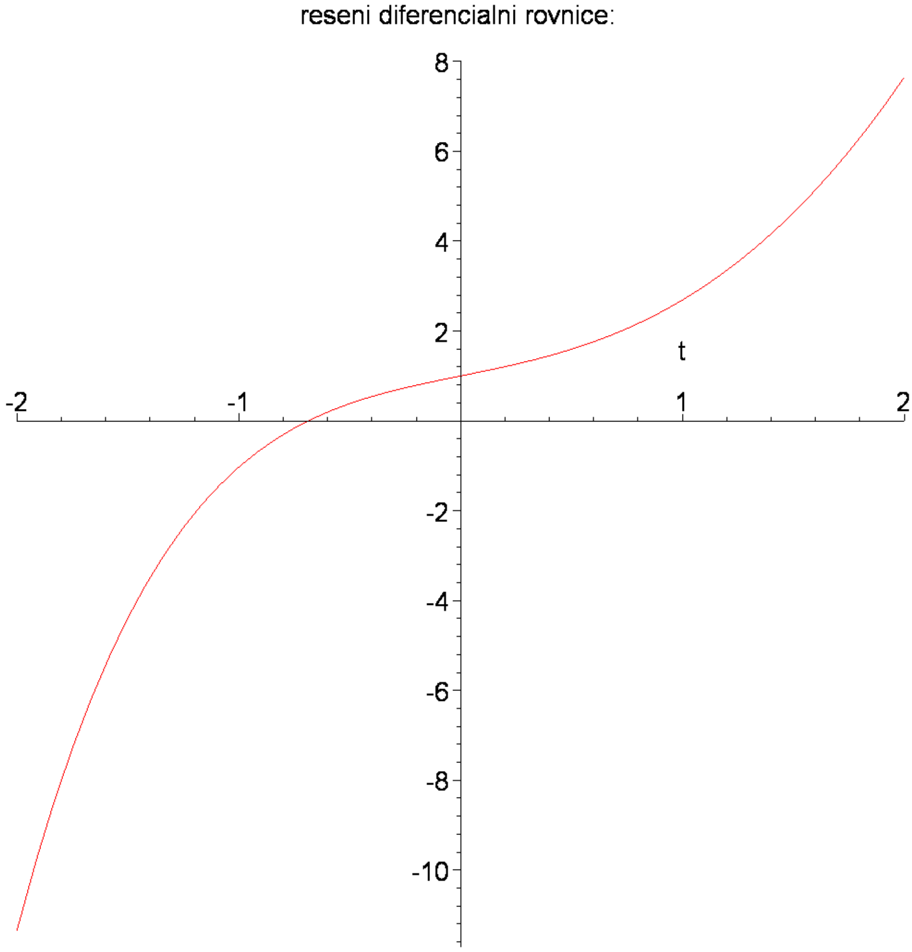
```
> lresult:=(2*s^4*y0+s^5*y0+2*s^3*dy0+s^4*dy0+ddy0*s^2+ddy0*s^3
+s^3*y0+dy0*s^2+6+6*s+s^2)/(s^3*(2*s+s^2+1));
```

$$lresult := \frac{6 + 3\,s^3 + s^4 + 3\,s^2 + 6\,s}{s^3\,(2\,s + s^2 + 1)}$$

```
> dresult:=invlaplace(lresult,s,t);
```

$$dresult := 9 - 6\,t + 3\,t^2 - \mathbf{e}^{(-t)}\,(8+t)$$

```
> plot(dresult,t=-2..2,title=`reseni diferencialni rovnice:`);
```



Diferenciální rovnici vzhledem k počátečním podmínkám řeší rovnice: y(t)=
$3\,t^2 - 6\,t + 9 - t\,\mathbf{e}^{(-t)} - 8\,\mathbf{e}^{(-t)}$

>

```
> lequation:=laplace((D@@3)(y)(t)-3*(D@@2)(y)(t)+3*D(y)(t)-y(t)
  =exp(t),t,s);
```

$$lequation := s^3\,\mathrm{laplace}(\mathrm{y}(t), t, s) - (\mathrm{D}^{(2)})(y)(0) - s\,\mathrm{D}(y)(0) - s^2\,\mathrm{y}(0) - 3\,s^2\,\mathrm{laplace}(\mathrm{y}(t), t, s) + 3\,\mathrm{D}(y)(0) + 3\,s\,\mathrm{y}(0) + 3\,s\,\mathrm{laplace}(\mathrm{y}(t), t, s) - 3\,\mathrm{y}(0) - \mathrm{laplace}(\mathrm{y}(t), t, s) = \frac{1}{s-1}$$

```
> lresult:=solve(lequation,laplace(y(t),t,s));
```

$$lresult := (1 + 6\,s\,\mathrm{y}(0) - 3\,\mathrm{y}(0) + (\mathrm{D}^{(2)})(y)(0)\,s - (\mathrm{D}^{(2)})(y)(0) + \mathrm{D}(y)(0)\,s^2 - 4\,s\,\mathrm{D}(y)(0) + s^3\,\mathrm{y}(0) - 4\,s^2\,\mathrm{y}(0) + 3\,\mathrm{D}(y)(0)) \Big/ ((s-1)\,(s^3 - 3\,s^2 + 3\,s - 1))$$

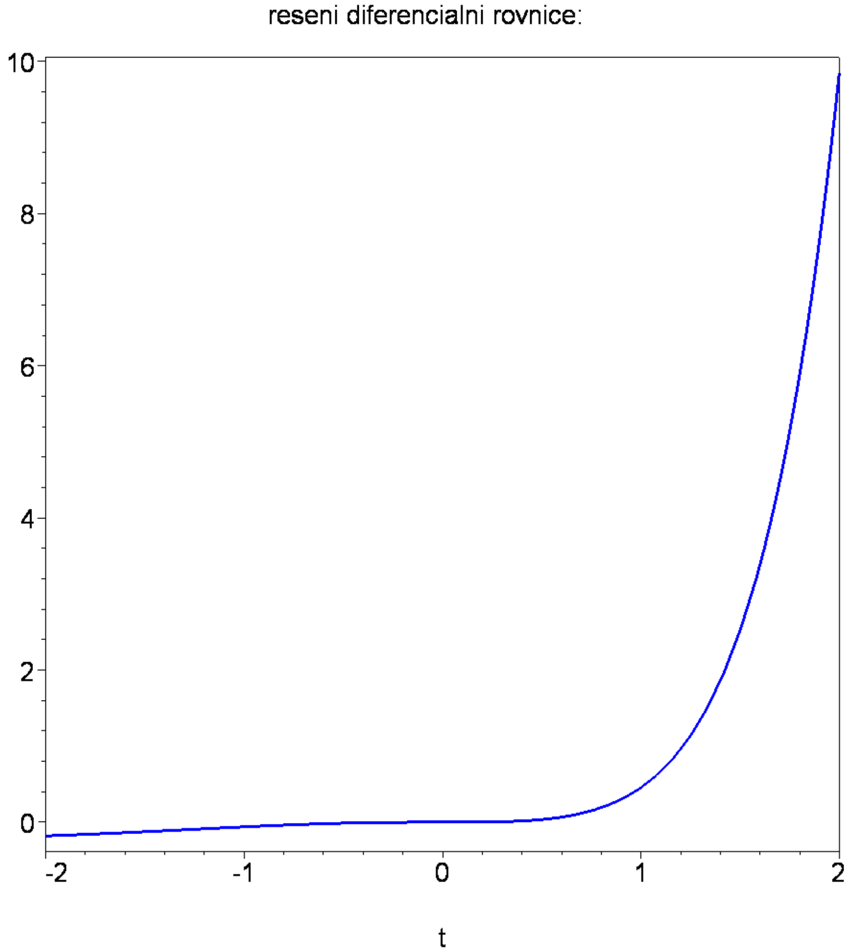
S využitím počátečních podmínek:

```
> dresult:=invlaplace(1/(s^4-4*s^3+6*s^2-4*s+1),s,t);
```

$$dresult := \frac{1}{6} t^3 e^t$$

Diferenciální rovnici řeší funkce y(t)= $\frac{1\, t^3\, e^t}{6}$

```
> plot(1/6*t^3*exp(t),t=-2..2,colour=blue,thickness=3,title=`re
  seni diferencialni rovnice:`,axes=boxed);
```




```
>
```

```
> reseni:=dsolve(diff(y(x),x)+2*y(x)=sin(3*x)-exp(-x));
```

$$reseni := y(x) = -\frac{3}{13}\cos(3\,x) + \frac{2}{13}\sin(3\,x) - e^{(-x)} + e^{(-2\,x)}\,\_C1$$

```
> lequation:=laplace(diff(y(x),x)+2*y(x)=sin(3*x)-exp(-x),x,s);
```

$$lequation := s\ \mathrm{laplace}(y(x), x, s) - y(0) + 2\ \mathrm{laplace}(y(x), x, s) = \frac{3}{s^2+9} - \frac{1}{1+s}$$

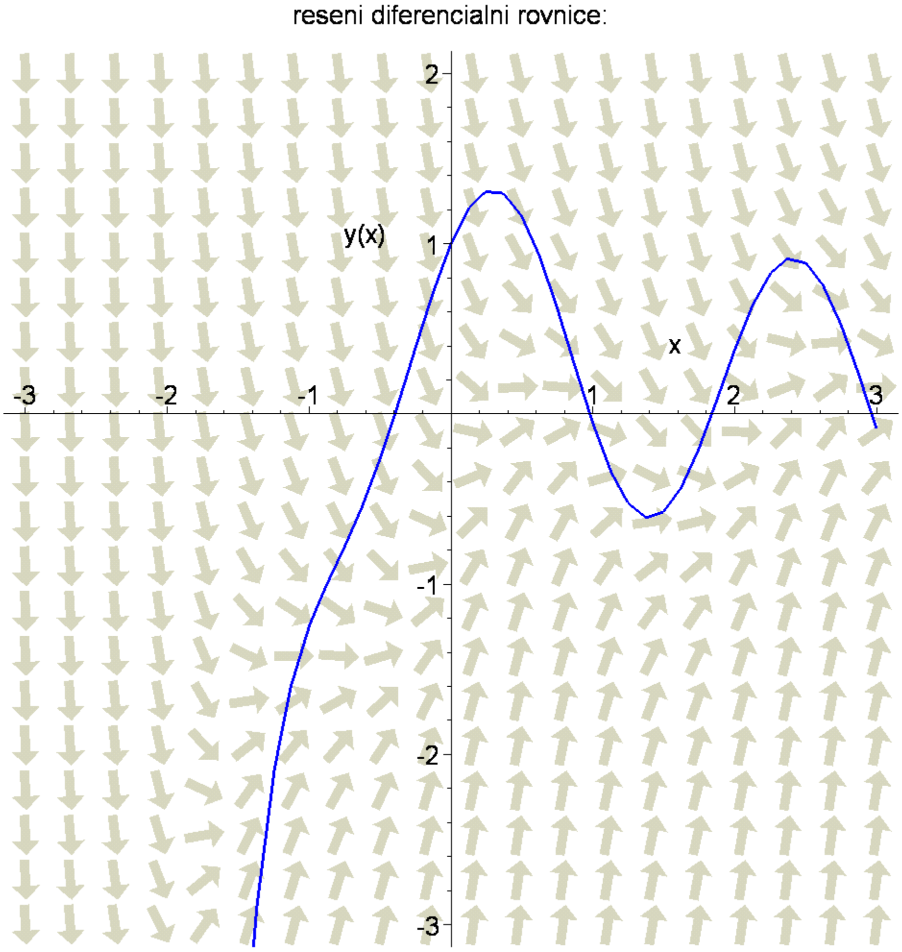
```
> solve(lequation,laplace(y(t),t,s));
```

$$(1+6\,s\,y(0)-3\,y(0)+(D^{(2)})(y)(0)\,s-(D^{(2)})(y)(0)+D(y)(0)\,s^2-4\,s\,D(y)(0)+s^3\,y(0)-4\,s^2\,y(0)+3\,D(y)(0))\,/\,((s-1)\,(s^3-3\,s^2+3\,s-1))$$

```
> dresult:=invlaplace(s*(-s^2-s^3-9-9*s-6+3*s-s^2)/(3*s^3+s^4+27*s+11*s^2+18),s,t);
```

$$dresult := -\mathrm{Dirac}(t)+\mathbf{e}^{(-t)}+\frac{6}{13}\cos(3\,t)+\frac{9}{13}\sin(3\,t)-\frac{6}{13}\,\mathbf{e}^{(-2\,t)}$$

```
> pole:=dfieldplot(diff(y(x),x)+2*y(x)=sin(3*x)-exp(-x),y(x),x=-3..3,y=-3..2,arrows=THICK,colour=wheat):
> krivka:=plot(dresult,t=-3..3,colour=blue,thickness=3):
> display(pole,krivka,title=`reseni diferencialni rovnice:`);
```




```
>
> lequation:=laplace(4*diff(y(t),t)-y(t)=exp(2*t)*cos(2*t),t,s);
```

lequation :=

$$4\,s\,\mathrm{laplace}(\mathrm{y}(t),t,s)-4\,\mathrm{y}(0)-\mathrm{laplace}(\mathrm{y}(t),t,s)=\frac{s-2}{(s-2-2\,I)\,(s-2+2\,I)}$$

```
> y(0):=2;
```

$$\mathrm{y}(0):=2$$

```
> lresult:=solve(lequation,laplace(y(t),t,s));
```

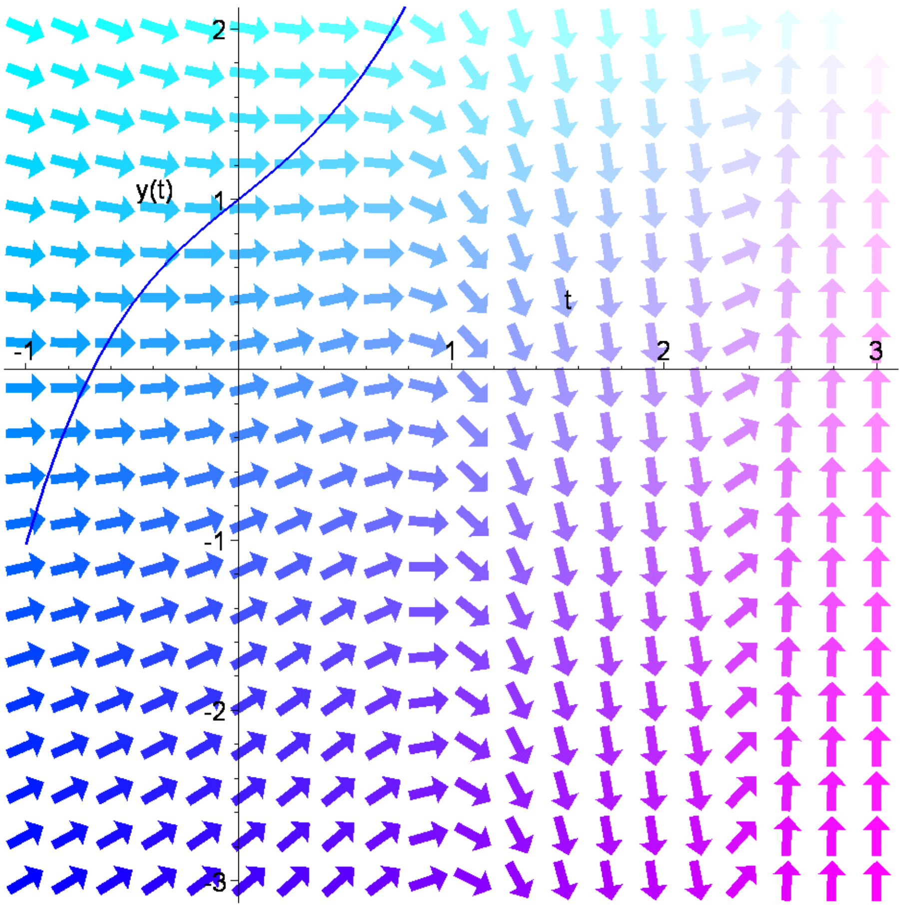
$$lresult:=(1+6\,s\,\mathrm{y}(0)-3\,\mathrm{y}(0)+(\mathrm{D}^{(2)})(y)(0)\,s-(\mathrm{D}^{(2)})(y)(0)+\mathrm{D}(y)(0)\,s^2$$
$$-4\,s\,\mathrm{D}(y)(0)+s^3\,\mathrm{y}(0)-4\,s^2\,\mathrm{y}(0)+3\,\mathrm{D}(y)(0))\Big/((s-1)\,(s^3-3\,s^2+3\,s-1))$$

```
> dresult:=invlaplace(lresult,s,t);
```

$$dresult:=9-6\,t+3\,t^2-\mathbf{e}^{(-t)}\,(8+t)$$

```
> pole:=dfieldplot(4*diff(y(t),t)+y(t)=exp(2*t)*cos(2*t),y(t),t
  =-1..3,y=-3..2,arrows=THICK,colour=[t/3,y/3,1]):
> krivka:=plot(dresult,t=-1..3,colour=blue,thickness=3):
> display(pole,krivka,title=`reseni diferencialni rovnice:`);
```

reseni diferencialni rovnice:




```
>
```

Nejprve zkusíme vyřešit rovnici klasickou cestou:

```
> dresult:=dsolve(diff(y(t),t)-5*y(t)=(cos(3*t))^2,y(t));
```

$$dresult := y(t) = -\frac{5}{122}\cos(6\,t) + \frac{3}{61}\sin(6\,t) - \frac{1}{10} + e^{(5\,t)}\,\_C1$$

Sestrojíme Laplaceův obraz:

```
> lequation:=laplace(diff(y(t),t)-5*y(t)=(cos(3*t))^2,t,s);
```

$$lequation := s\,\mathrm{laplace}(y(t), t, s) - y(0) - 5\,\mathrm{laplace}(y(t), t, s) = \frac{18 + s^2}{(s^2 + 36)\,s}$$

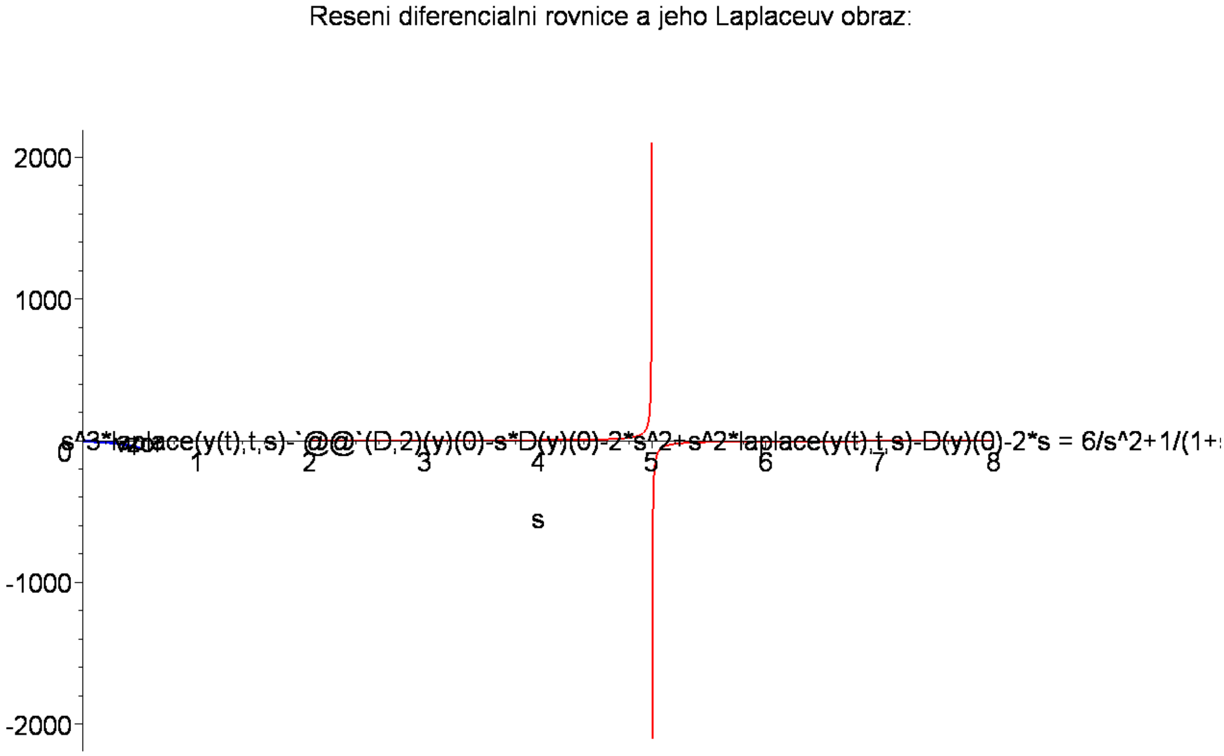
```
> lresult:=solve(s*laplace(y(t),t,s)-y(0)-5*laplace(y(t),t,s) =
  18*(1+1/18*s^2)/(s*(s^2+36)),laplace(y(t),t,s));
```

$$lresult := \frac{18 + s^2 + s^3\,y(0) + 36\,s\,y(0)}{s\,(s^2 + 36)\,(s - 5)}$$

```
> dresult:=invlaplace(lresult,s,t);
```

$$dresult := 9 - 6\,t + 3\,t^2 - e^{(-t)}\,(8 + t)$$

```
> pl1:=plot(-(s^3+36*s-18-s^2)/(s^3+36*s-5*s^2-180),s=2..5,colo
  ur=red,thickness=2):
> pl2:=plot(-(s^3+36*s-18-s^2)/(s^3+36*s-5*s^2-180),s=5..8,colo
  ur=red,thickness=2):
> pl3:=plot(-Dirac(t)-262/61*exp(5*t)+18/61*cos(6*t)+15/61*sin(
  6*t),t=0.001..0.55,colour=blue,thickness=3):
> tx1:=textplot([5,10,`obraz`]):
> tx2:=textplot([0.5,-10,`vzor`]):
> display(pl1,pl2,pl3,tx1,tx2,title=`Reseni diferencialni
  rovnice a jeho Laplaceuv obraz:`);
```



```
> dequation:=sum((-1)^i*diff(y(t),t$i), i=0..n)=1;
```

$$dequation := \sum_{i=0}^{n} (-1)^i \left(\frac{d^i}{dt^i} y(t) \right) = 1$$

```
> lequation:=sum((-2)^i*s^i*laplace(y(t),t,s), i=0..n)=1/s;
```

$$lequation := -\frac{\text{laplace}(y(t), t, s)\,(-2\,s)^{(n+1)}}{2\,s+1} + \frac{\text{laplace}(y(t), t, s)}{2\,s+1} = \frac{1}{s}$$

```
> lresult:=solve(lequation,laplace(y(t),t,s));
```

$$lresult := (1 + 6\,s\,y(0) - 3\,y(0) + (D^{(2)})(y)(0)\,s - (D^{(2)})(y)(0) + D(y)(0)\,s^2 - 4\,s\,D(y)(0) + s^3\,y(0) - 4\,s^2\,y(0) + 3\,D(y)(0)) \Big/ ((s-1)\,(s^3 - 3\,s^2 + 3\,s - 1))$$

Laplaceův obraz pro funkci f(x)=1 je jednoduchý L(f)=1/s. Výraz $-1 + 2\,s - 4\,s^2 + 8\,s^3$ zkusíme rozložit na součin.

```
> factor(-1+2*s-4*s^2+8*s^3);
```

$$(2\,s-1)\,(4\,s^2+1)$$

```
> dresult:=invlaplace(lresult,s,t);
```

$$dresult := 9 - 6\,t + 3\,t^2 - \mathbf{e}^{(-t)}\,(8+t)$$

```
> dresult:=invlaplace(lresult,s,t);
```

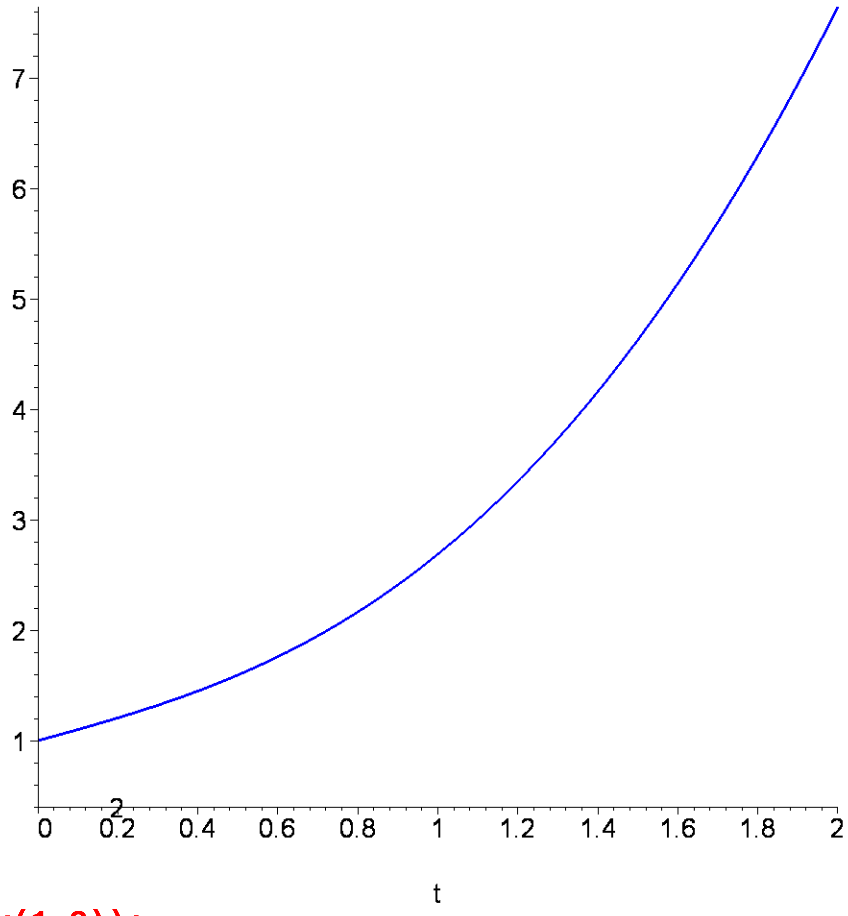
$$dresult := 9 - 6\,t + 3\,t^2 - \mathbf{e}^{(-t)}\,(8+t)$$

```
> fx:=unapply(dresult,t,n);
```

$$fx := (t, n) \rightarrow 9 - 6\,t + 3\,t^2 - \mathbf{e}^{(-t)}\,(8+t)$$

```
> sekvence:=seq(display(plot(fx(t,n),t=0..2,colour=blue,thickne
  ss=3,title=`ahoj`),textplot([0.2,0.4,convert(n,string)])),n=2
  ..10):
> display(sekvence,insequence=true,title=`animace posloupnosti
  reseni rovnic pro stoupajici n:`);
```

animace posloupnosti reseni rovnic pro stoupajici n:




```
> evalf(fx(1,3));
```

$$2.689085029$$

```
>
>
> factor(-1+2*s-4*s^2+8*s^3);
```

$$(2\,s-1)\,(4\,s^2+1)$$

```
>
> f:=1/(x^7-7*x^6+17*x^5-55*x^4+79*x^3+87*x^2+63*x+135):
> laplace(f,x,s);
```

$$\operatorname{laplace}\left(\frac{1}{(x-3)\,(x-5)\,(x+1)\,(x^2+9)\,(x^2+1)}, x, s\right)$$

Jak se zdá, Maple se neumí poradit s tak komplikovanou racionální funkcí. Rozložíme funkci na parciální zlomky:

```
> convert(f,parfrac,x);
```

$$\frac{1}{480\,(x+1)}+\frac{1}{10608\,(x-5)}-\frac{1}{1440\,(x-3)}+\frac{11-3\,x}{2080\,(x^2+1)}+\frac{-39-x}{24480\,(x^2+9)}$$

Rozložili jsme funkci na součet několika jednodušších.

```
> laplace(1/(x-3),x,s);
```

$$e^{(-3\,s)}\,(\mathrm{Ei}(1,-3\,s)+\ln(-s)-\ln(s))$$

```
> lf1:=s->1/1440*exp(-3*s)*Ei(3*s):
> laplace(1/(x-5),x,s);
```

$$e^{(-5\,s)}\,(\mathrm{Ei}(1,-5\,s)+\ln(-s)-\ln(s))$$

```
> lf2:=s->-1/10608*exp(-5*s)*Ei(5*s):
> laplace(1/(x+1),x,s);
```

$$e^{s}\,\mathrm{Ei}(1,s)$$

```
> lf3:=s->-1/480*exp(s)*Ei(-s):
> laplace((39+x)/(x^2+9),x,s);
```

$$\left(\frac{1}{2}-\frac{13}{2}I\right)\mathrm{Ei}(1,-3\,I\,s)\,e^{(-3\,I\,s)}+\left(\frac{1}{2}+\frac{13}{2}I\right)\mathrm{Ei}(1,3\,I\,s)\,e^{(3\,I\,s)}$$

```
> lf4:=s->-1/24480*(-(cos(sqrt(9)*s)-13/3*sqrt(9)*sin(sqrt(9)*s
  ))*Ci(sqrt(9)*s)-(sin(sqrt(9)*s)+13/3*sqrt(9)*cos(sqrt(9)*s))
  *Ssi(sqrt(9)*s)):
> laplace((-11+3*x)/(x^2+1),x,s);
```

$$\left(\frac{3}{2}+\frac{11}{2}I\right)\mathrm{Ei}(1,-I\,s)\,e^{(-I\,s)}+\left(\frac{3}{2}-\frac{11}{2}I\right)\mathrm{Ei}(1,s\,I)\,e^{(s\,I)}$$

```
> lf5:=s->-1/2080*(-(3*cos(s)+11*sin(s))*Ci(s)-(3*sin(s)-11*cos
  (s))*Ssi(s)):
```

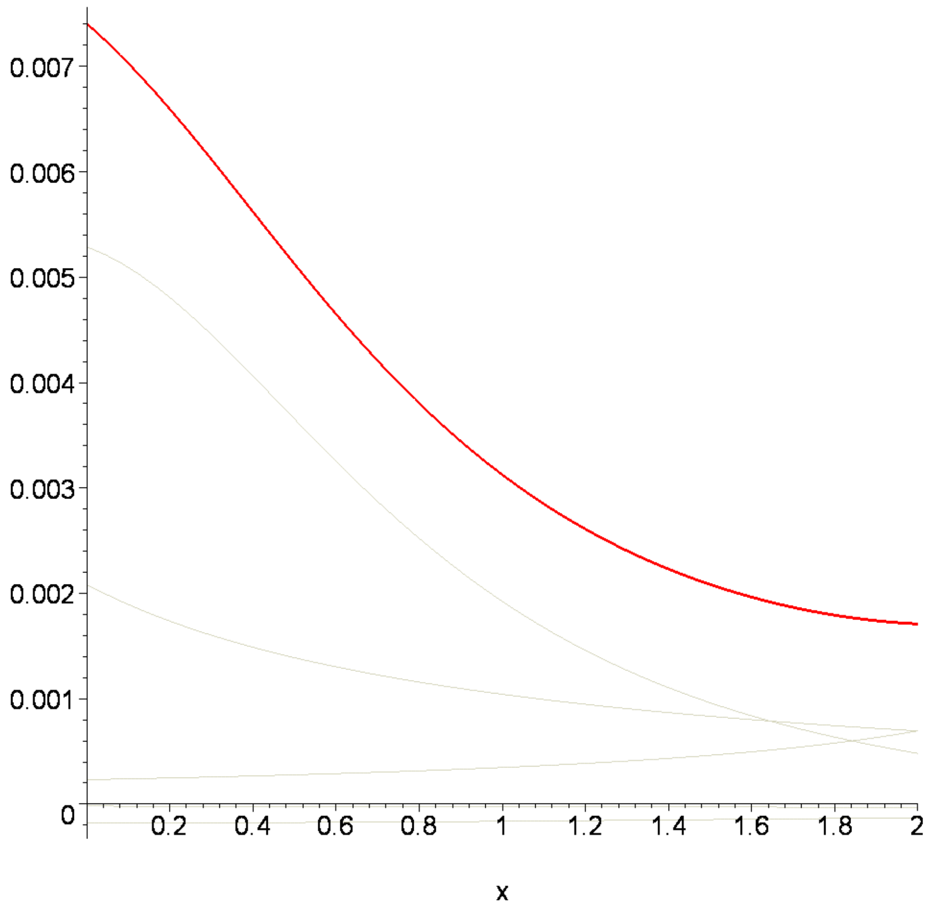
Laplaceův obraz funkce získáme jako součet obrazů jednodušších funkcí:

```
> lf:=s->lf1(s)+lf2(s)+lf3(s)+lf4(s)+lf5(s);
```

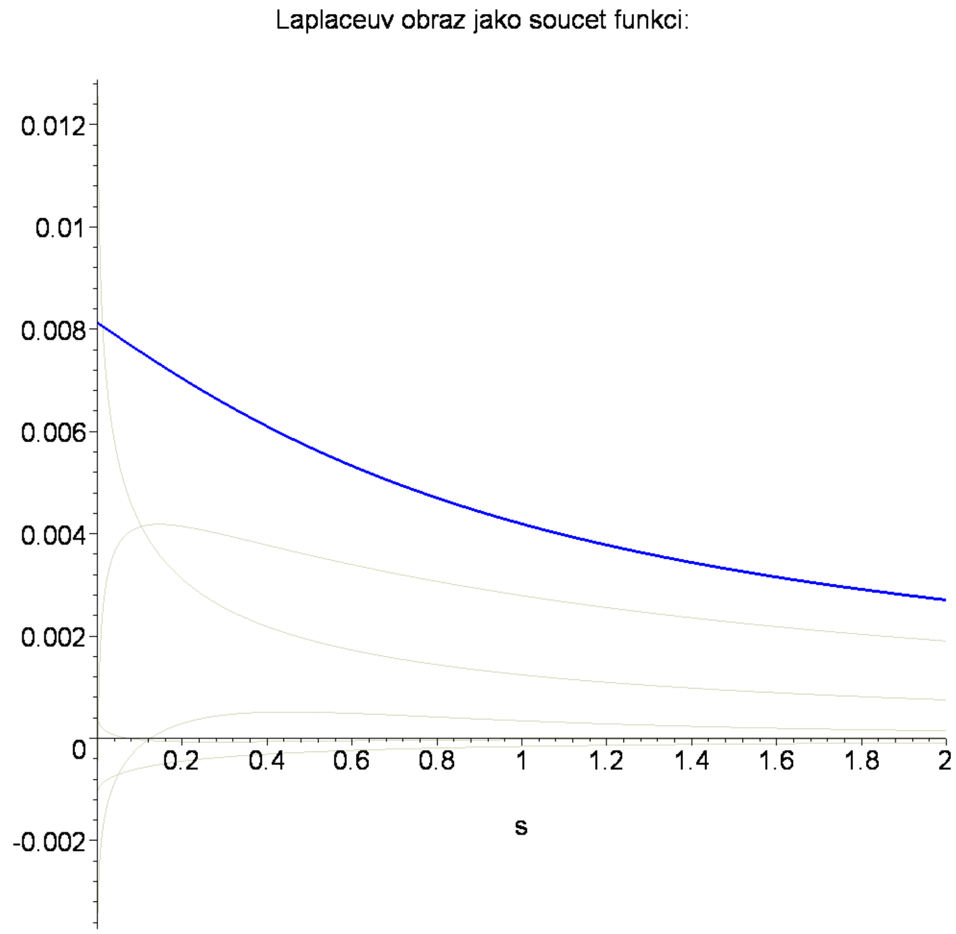
$$lf := s \rightarrow \mathrm{lf1}(s)+\mathrm{lf2}(s)+\mathrm{lf3}(s)+\mathrm{lf4}(s)+\mathrm{lf5}(s)$$

```
> f:=x->1/(x^7-7*x^6+17*x^5-55*x^4+79*x^3+87*x^2+63*x+135):
> f1:=x->-1/1440*1/(x-3): f2:=x->1/10608*1/(x-5):
  f3:=x->1/480*1/(x+1): f4:=x->-1/24480*(39+x)/(x^2+9):
  f5:=x->-1/2080*(-11+3*x)/(x^2+1):
> main:=plot(f(x),x=0..2,colour=red,thickness=3):
> snd:=plot({f1(x),f2(x),f3(x),f4(x),f5(x)},x=0..2,colour=wheat
  ):
> display(main,snd,title=`Vzor pro Laplaceovu transformaci jako
  soucet funkci:`);
```

Vzor pro Laplaceovu transformaci jako soucet funkci:




```
>
> lmain:=plot(lf(s),s=0..2,colour=blue,thickness=3):
> lnd:=plot({lf1(s),lf2(s),lf3(s),lf4(s),lf5(s)},s=0..2,colour=
  wheat):
> display(lmain,lnd,title=`Laplaceuv obraz jako soucet
  funkci:`);
```



Variační počet

```
>
> F:=y^2+4*y*y1+4*y1^2:
> Fy:=diff(F,y); Fyy1:=diff(Fy,y1); Fy1y1:=diff(diff(F,y1),y1);
  Fty1:=diff(diff(F,t),y1);
```

$$Fy := 2\,y + 4\,y1$$

$$Fyy1 := 4$$

$$Fy1y1 := 8$$

$$Fty1 := 0$$

Eulerova rovnice pro F:

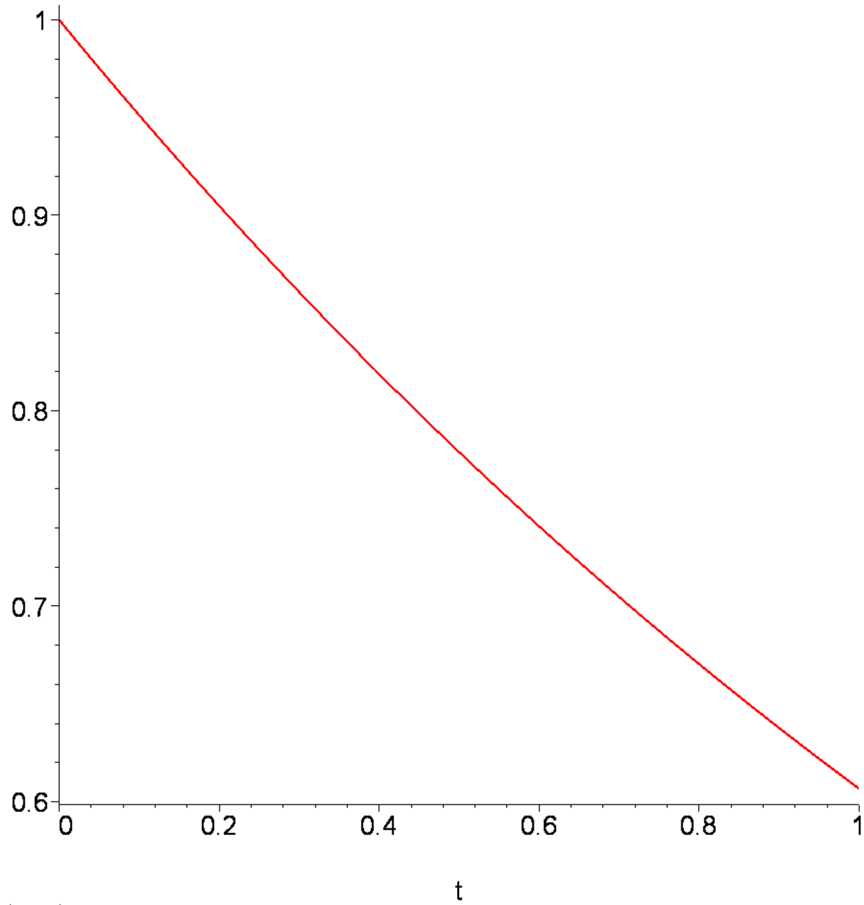
```
> dsolve(Fy1y1*diff(y(t),t)+Fyy1*y(t)+Fty1-2*y(t)-4*diff(y(t),t
  )=0,y(t));
```

$$y(t) = \_C1\, \mathbf{e}^{\left(-\frac{t}{2}\right)}$$

Pro splnění okrajových podmínek je konstanta c_1 rovna 1.

```
> plot(exp(-1/2*t),t=0..1,colour=red,thickness=3,title=`reseni
  optimalizace:`);
```

reseni optimalizace:



funkce y(t)=$\mathbf{e}^{\left(-\frac{1\,t}{2}\right)}$ minimalizuje V[y].

>

```
> F:=2*y*e^t+y^2-y1^2;
```

$$F := 2\,y\,e^t + y^2 - y1^2$$

```
> Fy:=diff(F,y); Fyy1:=diff(Fy,y1); Fy1y1:=diff(diff(F,y1),y1);
Fty1:=diff(diff(F,t),y1);
```

$$Fy := 2\,e^t + 2\,y$$

$$Fyy1 := 0$$

$$Fy1y1 := -2$$

$$Fty1 := 0$$

```
> eeq:=-2*diff(y(t),t$2)-2*exp(t)-2*y(t)=0;
```

$$eeq := -2\left(\frac{d^2}{dt^2}\,y(t)\right) - 2\,\mathbf{e}^t - 2\,y(t) = 0$$

```
> vysledek:=dsolve(eeq,y(t));
```

$$vysledek := y(t) = \sin(t)\ \_C2 + \cos(t)\ \_C1 - \frac{1}{2}\mathbf{e}^{t}$$

```
> y:=t->(-1/2*cos(t)*exp(t)-1/2*sin(t)*exp(t))*sin(t)+(-1/2*cos
  (t)*exp(t)+1/2*sin(t)*exp(t))*cos(t)+_C1*sin(t)+_C2*cos(t);
```

$$y := t \to \left(-\frac{1}{2}\cos(t)\ \mathbf{e}^{t} - \frac{1}{2}\sin(t)\ \mathbf{e}^{t}\right)\sin(t) + \left(-\frac{1}{2}\cos(t)\ \mathbf{e}^{t} + \frac{1}{2}\sin(t)\ \mathbf{e}^{t}\right)\cos(t) + \_C1\ \sin(t) + \_C2\ \cos(t)$$

```
> evalf(y(0));
```

$$-0.5000000000 + \_C2$$

Z toho plyne, že c2=3/2

```
> c2:=3/2:
> evalf(y(2));
```

$$-3.694528049 + 0.9092974268\ \_C1 - 0.4161468365\ \_C2$$

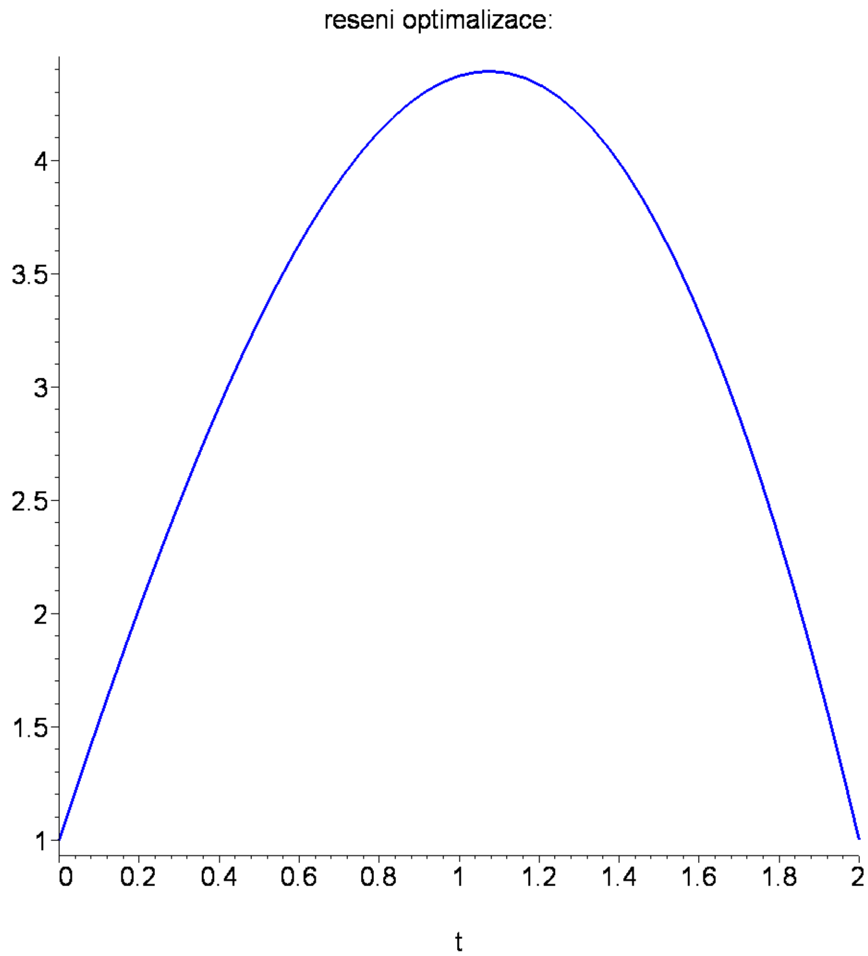
```
> c1:=solve(-3.694528050+.9092974268*c1-.4161468365*3/2=1,c1);
```

$$c1 := 5.849294354$$

```
>
> y(t):=t->(-1/2*cos(t)*exp(t)-1/2*sin(t)*exp(t))*sin(t)+(-1/2*
  cos(t)*exp(t)+1/2*sin(t)*exp(t))*cos(t)+5.849294354*sin(t)+3/
  2*cos(t);
```

$$y(t) := t \to \left(-\frac{1}{2}\cos(t)\ \mathbf{e}^{t} - \frac{1}{2}\sin(t)\ \mathbf{e}^{t}\right)\sin(t) + \left(-\frac{1}{2}\cos(t)\ \mathbf{e}^{t} + \frac{1}{2}\sin(t)\ \mathbf{e}^{t}\right)\cos(t) + 5.849294354\ \sin(t) + \frac{3}{2}\cos(t)$$

```
> plot((-1/2*cos(t)*exp(t)-1/2*sin(t)*exp(t))*sin(t)+(-1/2*cos(
  t)*exp(t)+1/2*sin(t)*exp(t))*cos(t)+5.849294354*sin(t)+3/2*co
  s(t),t=0..2,title=`reseni
  optimalizace:`,colour=blue,thickness=3);
```


reseni optimalizace:
4
3.5
3
2.5
2
1.5
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
t

>