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A Simple Proof of Cauchy Theorem

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The article presents one of the simplest proofs of the most fundamental theorem of complex analysis requiring, besides the Cauchy Theorem and Formula for rectangles, only some well-known and easy to prove properties of curvilinear integrals and of the index of a point with respect to a cycle.

In what follows, a more detailed proof including all auxiliary assertions will be presented. However, *the general idea of the present proof and its particular steps remain exactly the same as in the above article.* At the end of the article, Czech readers will find commentaries to the auxiliary assertions together with the corresponding Czech sources.

A Simple Proof of the Cauchy Integral Theorem for Cycles

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Assuming the validity of the Cauchy Theorem and the Cauchy Formula for Rectangles (see [3], Theorems 6.1.1. and 6.1.2, p.104), we shall prove the following assertion:

Cauchy's Theorem for Cycles. *If Γ is a cycle nullhomologous in an open set $\Omega \subset \mathbb{C}$ and F a function holomorphic in Ω, then $\int_\Gamma F = 0$.*

First let us explain the necessary notions and notation. $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Z}$ denote the set of all (finite) **complex numbers**, (finite) **real numbers**, and **integers**, respectively. Let $\operatorname{Re} z$ and $\operatorname{Im} z$ be the **real** and **imaginary parts** of $z \in \mathbb{C}$, respectively. For any $M \subset \mathbb{C}$, let $\operatorname{int} M$ and ∂M denote the **interior** and the **boundary** of M, respectively.

A **curve** is a continuous mapping φ of an interval $\langle \alpha, \beta \rangle \subset \mathbb{R}$ into $\mathbb{C}$ for which the set of all sums $\sum_{k=1}^{n} |\varphi(t_k) - \varphi(t_{k-1})|$ where $\alpha = t_0 < t_1 < \ldots < t_n = \beta$ is bounded above; the supremum of all such sums is called **length** of φ and denoted by $L(\varphi)$. The set $\langle \varphi \rangle := \varphi(\langle \alpha, \beta \rangle)$ is the **geometric image** of φ.

For any function F continuous on $\langle \varphi \rangle$, the **curvilinear integral** $\int_\varphi F$ is defined as the Stieltjes integral $\int_\alpha^\beta (F \circ \varphi)\, d\varphi$. The curve φ is called **closed**, if $\varphi(\alpha) = \varphi(\beta)$.

If φ is a closed curve and $\zeta \in \mathbb{C} - \langle\varphi\rangle$, then the number

$$\operatorname{ind}_\varphi \zeta := \int_\varphi \frac{dz}{z-\zeta}$$

is called **index** of ζ with respect to φ.

A **cycle** in Ω is any non-empty finite system $\Gamma = \{\varphi_1, \varphi_2, \dots, \varphi_n\}$ of closed curves in Ω, $\langle\Gamma\rangle := \bigcup_{k=1}^n \langle\varphi_k\rangle$ being its **geometric image**. Set $L(\Gamma) := \sum_{k=1}^n L(\varphi_k)$, $\int_\Gamma F := \sum_{k=1}^n \int_{\varphi_k} F$ for any continuous $F : \langle\Gamma\rangle \to \mathbb{C}$, and $\operatorname{ind}_\Gamma \zeta := \sum_{k=1}^n \operatorname{ind}_{\varphi_k} \zeta$ for all $\zeta \in \mathbb{C} - \langle\Gamma\rangle$. Γ is called **nullhomologous** in Ω, if $\operatorname{ind}_\Gamma \zeta = 0$ for all $z \in \mathbb{C} - \Omega$.

By a **rectangle** we mean any set of the form

$$Q = \{z \in \mathbb{C};\ \alpha \le \operatorname{Re} z \le \beta,\ \gamma \le \operatorname{Im} z \le \delta\} \tag{1}$$

where $\langle\alpha, \beta\rangle \subset \mathbb{R}$, $\langle\gamma, \delta\rangle \subset \mathbb{R}$. Points $V_0 = V_4 := \alpha + i\beta$, $V_1 := \beta + i\gamma$, $V_2 := \beta + i\delta$, and $V_3 := \alpha + i\delta$ are the **vertices** of Q, $\lambda_k(t) := V_{k-1} + t(V_k - V_{k-1})$, $t \in \langle 0, 1\rangle$, $k = 1, \dots, 4$, its **oriented sides**, the curve (Q) defined by setting

$$(Q)(t) := \left\{ \begin{array}{ll} \lambda_1(t) & \text{for } t \in \langle 0,1\rangle \\ \lambda_2(t-1) & \text{for } t \in \langle 1,2\rangle \\ \lambda_3(t-2) & \text{for } t \in \langle 2,3\rangle \\ \lambda_4(t-3) & \text{for } t \in \langle 3,4\rangle \end{array} \right\} \tag{2}$$

being its **oriented boundary**. Sets $\langle\lambda_k\rangle$, $k = 1, \dots, 4$, are (unoriented) **sides** of Q.

In what follows we shall need the following well-known assertions:

Lemma 1. *For any rectangle Q,*

$$\operatorname{ind}_{(Q)} \zeta = \left\{ \begin{array}{ll} 1 & \text{for all } \zeta \in \operatorname{int} Q \\ 0 & \text{for all } \zeta \in \mathbb{C} - Q \end{array} \right\} \tag{3}$$

holds. (See [3], Theorem 3.2.4, p.55 and Exercise 3.2.9, p.57.)

Lemma 2. *If Q, Q^* are rectangles satisfying $\operatorname{int} Q \cap \operatorname{int} Q^* = \emptyset$ and if oriented sides λ, λ^* of Q, Q^*, respectively, satisfy $\langle\lambda\rangle = \langle\lambda^*\rangle$, then $\int_\lambda f + \int_{\lambda^*} f = 0$ for any continuous function $f : \langle\lambda\rangle \to \mathbb{C}$.* (Cf. with [3], Exercise 6.1.1, p.102.)

Lemma 3. *If Γ is a cycle, then $\operatorname{ind}_\Gamma$ is an integer-valued function continuous in its definition domain $\mathbb{C} - \langle\Gamma\rangle$ and constant in every component of the set $\mathbb{C} - \langle\Gamma\rangle$; in the unbounded component of $\mathbb{C} - \langle\Gamma\rangle$, $\operatorname{ind}_\Gamma$ vanishes.* (See [3], p.112.)

Lemma 4. *Let φ be a curve, suppose the set $M \subset \mathbb{C}$ is either open or closed and let $f : \langle\varphi\rangle \times M \to \mathbb{C}$ be a continuous function. Then the integral $I(\zeta) := \int_\varphi f(z, \zeta)\, dz$ is continuous in M.* (See [3], Theorem 5.2.1, p.86.)

Lemma 5. *If Γ is a cycle and if a continuous function $f : \langle\Gamma\rangle \to \mathbb{C}$ satisfies $|f| \le K < +\infty$ in $\langle\Gamma\rangle$, then $|\int_\Gamma f| \le K\, L(\Gamma)$.* (Cf. with [3], (7), p.85.)

Lemma 6. *If $\varphi : \langle \alpha, \beta \rangle \to \mathbb{C}$ is a curve, $M \subset \mathbb{C}$ a compact set and if the function $f(z,\zeta)$ is continuous in $\langle \varphi \rangle \times M$, then for each $\varepsilon > 0$ there is a finite set $\{z_1, \ldots, z_r\} \subset \langle \varphi \rangle$ satisfying $| \int_\varphi f(z,\zeta) - \sum_{k=1}^{n} f(z_k,\zeta)(z_k - z_{k-1})| < \varepsilon$ for all $\zeta \in M$.* (See [3], Theorem 5.2.1, p.86.)

Cauchy's Theorem and Formula for Rectangles. *If $\Omega \subset \mathbb{C}$ is an open set containing a rectangle Q and if F is holomorphic in Ω, then $\int_{(Q)} F = 0$ and the identity*

$$\text{(4)} \qquad \int_{(Q)} \frac{F(z)}{z-\zeta}\, dz = 2\pi i\, F(\zeta)\, \mathrm{ind}_{(Q)}\, \zeta \quad \textit{holds for all } \zeta \in \Omega - \partial Q .$$

P r o o f of the Cauchy Theorem for Cycles. Since Γ is nullhomologous in Ω, the set

$$\text{(5)} \qquad N := \{z \in \mathbb{C};\ \mathrm{ind}_\Gamma\, z \neq 0\} \cup \langle \Gamma \rangle$$

is a subset of Ω. It is closed, because $\mathbb{C} - N$ is identical with the union of all components of $\mathbb{C} - \langle \Gamma \rangle$ with $\mathrm{ind}_\Gamma = 0$ and, therefore, is open. As $\mathrm{ind}_\Gamma\, \zeta = 0$ for all ζ with sufficiently large $|\zeta|$, the set N is bounded. Hence N is a compact subset of Ω and there exists a $\delta > 0$ satisfying

$$\text{(6)} \qquad \mathrm{dist}(z, N) < 2\delta \ \Rightarrow\ z \in \Omega.$$

Let $\mathcal{S}_0$ be the system of all squares

$$\text{(7)} \qquad Q_{mn} := \{z;\ (m-1)\delta \le \mathrm{Re}\, z \le m\delta,\ (n-1)\delta \le \mathrm{Im}\, z \le n\delta\}$$

where $m \in \mathbb{Z}$, $n \in \mathbb{Z}$. Let $\mathcal{S}$ be the system of all squares $Q \in \mathcal{S}_0$ with $Q \cap N \neq \emptyset$. Since N is bounded, $\mathcal{S}$ is finite, and, by (6),

$$\text{(8)} \qquad N \subset \bigcup_{Q \in \mathcal{S}} Q \subset \Omega$$

holds. Denote by $\mathcal{T}$ the set of all oriented sides of all squares $Q \in \mathcal{S}$.

Let the system $\mathcal{T}_1$ contain exactly all curves $\lambda \in \mathcal{T}$ for which $\langle \lambda \rangle$ is a side of exactly one square $Q \in \mathcal{S}$, and set $\mathcal{T}_2 := \mathcal{T} - \mathcal{T}_1$; thus $\mathcal{T}_2$ consists of exactly all curves $\lambda \in \mathcal{T}$ for which $\langle \lambda \rangle$ is a side of two different squares from $\mathcal{S}$. According to Lemma 2,

$$\text{(9)} \qquad \sum_{Q \in \mathcal{S}} \int_{(Q)} f = \sum_{\lambda \in \mathcal{T}} \int_\lambda f = \sum_{\lambda \in \mathcal{T}_1} \int_\lambda f + \sum_{\lambda \in \mathcal{T}_2} \int_\lambda f = \sum_{\lambda \in \mathcal{T}_1} \int_\lambda f$$

holds for any continuous function $f : \bigcup_{Q \in \mathcal{S}} \partial Q \to \mathbb{C}$.

According to (4), we have

$$F(\zeta) = \frac{1}{2\pi i} \sum_{Q \in \mathcal{S}} \int_{(Q)} \frac{F(z)}{z-\zeta}\, d\zeta \quad \text{for each} \quad \zeta \in \bigcup_{Q \in \mathcal{S}} \operatorname{int} Q\,. \tag{10}$$

For any $\zeta \in \bigcup_{Q \in \mathcal{S}} \operatorname{int} Q$, $F(z)/(z-\zeta)$ (as a function of variable z) is continuous in $\bigcup_{Q \in \mathcal{S}} \partial Q$; hence, by (9) and (10),

$$F(\zeta) = \frac{1}{2\pi i} \sum_{\lambda \in \mathcal{T}_1} \int_{\lambda} \frac{F(z)}{z-\zeta}\, d\zeta \quad \text{holds for all} \quad \zeta \in \bigcup_{Q \in \mathcal{S}} \operatorname{int} Q\,. \tag{11}$$

F is continuous in Ω and, by Lemma 4, the right-hand side of the last identity is continuous in $\mathbb{C} - \bigcup_{\lambda \in \mathcal{T}_1} \langle\lambda\rangle$; hence the identity

$$F(\zeta) = \frac{1}{2\pi i} \sum_{\lambda \in \mathcal{T}_1} \int_{\lambda} \frac{F(z)}{z-\zeta}\, d\zeta \quad \text{holds for each} \quad \zeta \in \bigcup_{Q \in \mathcal{S}} \operatorname{int} Q - \bigcup_{\lambda \in \mathcal{T}_1} \langle\lambda\rangle. \tag{12}$$

If λ is an oriented side of some square $Q \in \mathcal{S}_0$, then $\langle\lambda\rangle$ is a side of some other square $Q^* \in \mathcal{S}_0$ as well. If $\langle\lambda\rangle \cap N \neq \emptyset$, both squares Q, Q^* belong to $\mathcal{S}$ and, consequenty, $\lambda \in \mathcal{T}_2$; moreover, by (8), $Q \cup Q^* \subset \Omega$. So if $z \in \langle\lambda\rangle$ for some $\lambda \in \mathcal{T}_1$, then $z \in \Omega - N$. Hence, by definition of N, the following implication holds:

$$z \in \bigcup_{\lambda \in \mathcal{T}_1} \langle\lambda\rangle \;\Rightarrow\; \operatorname{ind}_{\Gamma} z = 0\,. \tag{13}$$

According to (5), (8), and (13), we have

$$\langle\Gamma\rangle \subset \bigcup_{Q \in \mathcal{S}} Q - \bigcup_{\lambda \in \mathcal{T}_1} \langle\lambda\rangle\,, \tag{14}$$

so that, by (12),

$$F(\zeta) = \frac{1}{2\pi i} \sum_{\lambda \in \mathcal{T}_1} \int_{\lambda} \frac{F(z)}{z-\zeta}\, dz \quad \text{holds for each} \quad \zeta \in \langle\Gamma\rangle. \tag{15}$$

Let $\varepsilon > 0$ be an arbitrary, but (for a moment) fixed number. Let p be the number of elements λ of the set $\mathcal{T}_1$ and fix (for a moment) a $\lambda \in \mathcal{T}_1$. By (14), the function $F(z)/(z-\zeta)$ of the two variables z, ζ is continuous on the compact set $\langle\lambda\rangle \times \langle\Gamma\rangle$. By Lemma 6, there exist points $z_1, \dots, z_r$ in $\langle\lambda\rangle$ such that the inequality

$$\Big| \int_{\lambda} \frac{F(z)}{z-\zeta}\, dz - \sum_{k=1}^{r} \frac{F(z_k)}{z_k-\zeta}\, (z_k - z_{k-1}) \Big| \le \frac{\varepsilon}{p} \quad \text{holds for each} \quad \zeta \in \langle\Gamma\rangle\,. \tag{16}$$

Further,

$$(17)\qquad \int_\Gamma \sum_{k=1}^{r} \frac{F(z_k)}{z_k-\zeta}\,(z_k-z_{k-1})\,d\zeta = \sum_{k=1}^{r} F(z_k)\,(z_k-z_{k-1}) \int_\Gamma \frac{d\zeta}{z_k-\zeta}$$
$$= \sum_{k=1}^{r} F(z_k)\,(z_k-z_{k-1})\cdot(-2\pi i \operatorname{ind}_\Gamma z_k) = 0$$

holds by (13). Consequently, relations

$$(18)\qquad \Big|\int_\Gamma \Big(\int_\lambda \frac{F(z)}{z-\zeta}\,dz\Big) d\zeta\Big|$$
$$= \Big|\int_\Gamma \Big(\int_\lambda \frac{F(z)}{z-\zeta}\,dz - \sum_{k=1}^{r} \frac{F(z_k)}{z_k-\zeta}\,(z_k-z_{k-1})\Big) d\zeta\Big| \le \frac{\varepsilon}{p}\,L(\Gamma)$$

hold for each $\lambda \in \mathcal{T}_1$ which (by (15)) implies the inequality

$$(19)\qquad \Big|\int_\Gamma F(\zeta)\,d\zeta\Big| = \Big|\int_\Gamma \Big(\frac{1}{2\pi i} \sum_{\lambda\in\mathcal{T}_1} \int_\lambda \frac{F(z)}{z-\zeta}\,dz\Big) d\zeta\Big| \le \frac{\varepsilon L(\Gamma)}{2\pi};$$

since $\varepsilon > 0$ was arbitrary, the left-hand side integral vanishes, Q.E.D.

* * *

Remark. By using the following Lemma 7 (instead of Lemma 6), the above proof may be shortened substantially.

Lemma 7. *If φ and ψ are curves and if $f : \langle\varphi\rangle \times \langle\psi\rangle \to \mathbb{C}$ is continuous, then*

$$(20)\qquad \int_\varphi \Big(\int_\psi f(z,\zeta)\,d\zeta\Big)\,dz = \int_\psi \Big(\int_\varphi f(z,\zeta)\,dz\Big)\,d\zeta\,.$$

An identity analogous to (20) *holds as well with φ or ψ replaced by a cycle.* (Cf. with [3], Theorem 5.2.7, p. 91.)

M o d i f i e d p r o o f of the Cauchy Theorem. (See [3], p. 115.) Proceed as above from the beginning up to (15) inclusive, and then continue as follows:

Since, by (14), $\langle\lambda\rangle \cap \Gamma = \emptyset$ holds for each $\lambda \in \mathcal{T}_1$, the function $F(z)/(z-\zeta)$ is continuous in $\langle\lambda\rangle \times \langle\Gamma\rangle$. By Lemma 7, we have

$$(21)\qquad \int_\Gamma F(\zeta)\,d\zeta = \frac{1}{2\pi i} \sum_{\lambda\in\mathcal{T}_1} \int_\Gamma \Big(\int_\lambda \frac{F(z)}{z-\zeta}\,dz\Big)\,d\zeta$$
$$= -\sum_{\lambda\in\mathcal{T}_1} \int_\lambda F(z)\Big(\frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{\zeta-z}\Big)\,dz = -\sum_{\lambda\in\mathcal{T}_1} \int_\lambda F(z) \operatorname{ind}_\Gamma z\,dz = 0\,,$$

because (13) implies $\operatorname{ind}_\Gamma z = 0$ for each $z \in \bigcup_{\lambda\in\mathcal{T}_1}\langle\lambda\rangle$.

This completes the modified proof.

Několik poznámek pro české čtenáře

K lemmatu 1: Nejjednodušší důkaz tohoto tvrzení lze provést pomocí Maříkovy věty [1]), jejíž aplikace je v tomto případě zcela jednoduchá:

Buď Q jako v (1) a nechť

$$A = \tfrac{1}{2}((\alpha+\beta)+(\gamma+\delta)i), \quad B = \tfrac{1}{2}((\alpha+3\beta)+(\gamma+\delta)i). \tag{22}$$

Protože úsečka s krajními body A, B protíná množinu $\langle\lambda_2\rangle$ v bodě $\lambda_2(\frac{1}{2})$ a protože $\operatorname{Im}\lambda_2$ v bodě $\frac{1}{2}$ roste, je $\operatorname{ind}_{(Q)} A = \operatorname{ind}_{(Q)} B + 1$. Protože B leží v neomezené komponentě množiny $\mathbb{C} - \partial Q$, je $\operatorname{ind}_{(Q)} B = 0$, takže $\operatorname{ind}_{(Q)} A = 1$. [2])

Čtenář, který ví, co jsou jednoznačné větve argumentu a jak souvisejí s indexem, může postupovat i přímo: Podle původní definice indexu [3]) je $\operatorname{ind}_{(Q)}\zeta = (A(4) - A(0))/(2\pi)$, kde A je jednoznačná větev $\arg\circ((Q)-\zeta)$ v intervalu $\langle 0,4\rangle$.

Je-li $\zeta \notin Q$, existuje (uzavřená) polopřímka P s krajním bodem ζ a disjunktní s Q. V množině $\mathbb{C} - P$ pak existuje jednoznačná větev $\arg(z-\zeta)$; označíme-li ji Ar, je funkce $A = \operatorname{Ar}\circ((Q)-\zeta)$ jednoznačnou větví $\arg\circ((Q)-\zeta)$ v intervalu $\langle 0,4\rangle$. Protože křivka (Q) je uzavřená, je její přírůstek na intervalu $\langle 0,4\rangle$ roven 0.

Je-li $\zeta \in \operatorname{int} Q$, označme

$$P_1 = \{z;\, \operatorname{Re} z \le \operatorname{Re}\zeta,\ \operatorname{Im} z = \operatorname{Im}\zeta\}, \quad P_2 = \{z;\, \operatorname{Re} z \ge \operatorname{Re}\zeta,\ \operatorname{Im} z = \operatorname{Im}\zeta\} \tag{23}$$

a nechť Ar_1 (resp. Ar_2) znamená jednoznačnou větev $\arg(z-\zeta)$ v $\mathbb{C} - P_1$, resp. v $\mathbb{C} - P_2$, která tam nabývá hodnot z intervalu $(-\pi,\pi)$ (resp. $(0,2\pi)$).

Pak je $A_1 = \operatorname{Ar}_1\circ((Q)-\zeta)$ (resp. $A_2 = \operatorname{Ar}_2\circ((Q)-\zeta)$)) jednoznačná větev $\arg\circ((Q)-\zeta)$ v intervalu $\langle 0,2\rangle$ (resp. $\langle 2,4\rangle$) a oba příslušné přírůstky jsou rovny π. Celkový přírůstek je tedy 2π a $\operatorname{ind}_{(Q)}\zeta$ se rovná 1.

K lemmatu 2: Pro vrcholy a orientované strany intervalu Q^* užívejme podobná označení jako u intervalu (1), přidejme jen hvězdičku. Je-li $\lambda = \lambda_1$ [4]), je $\lambda^* = \lambda_3^*$. (Protože úsečka $\langle\lambda_1\rangle$ je rovnoběžná s reálnou osou, není λ^* rovna ani λ_2^*, ani λ_4^*; protože $\operatorname{int} Q \cap \operatorname{int} Q^* = \emptyset$, není $\lambda^* = \lambda_1^*$.) Z toho plyne, že $\lambda(t) = V_0 + t(V_1 - V_0)$, $\lambda^*(t) = V_2^* + t(V_3^* - V_2^*)$ pro $t \in \langle 0,1\rangle$. Protože $V_0 = V_3^*$, $V_1 = V_2^*$, je

$$\lambda^*(t) = V_1 + t(V_0 - V_1) = tV_0 + (1-t)V_2 = \lambda(1-t). \tag{24}$$

Je-li funkce f spojitá v $\langle\lambda\rangle$, platí podle věty o převedení křivkového integrálu na Newtonův [5]) a podle věty o substituci $(1-t=\tau)$ rovnosti

$$\begin{aligned}\int_{\lambda^*} f &= \int_0^1 f(\lambda^*(t))(\lambda^*(t))'\,dt = \int_0^1 f(\lambda(1-t))(-\lambda'(1-t))\,dt \\ &= -\int_0^1 f(\lambda(\tau))\,\lambda'(\tau)\,d\tau = -\int_\lambda f.\end{aligned} \tag{25}$$

[1]) Viz [2], Věta 3,3,6, str. 93.
[2]) Srov. s Větou 3,3,5 na str. 92 v [2].
[3]) Definice v [2] na str. 88 je obecnější a nezávislá na křivkovém integrálu.
[4]) Pro ostatní strany intervalu Q je důkaz analogický.
[5]) Věta 6,1,6 z [2], str. 150

K lemmatu 3: Podrobnější informace o cyklech (včetně lemmatu 3, které je tam uvedeno jako tvrzení (7)) najde čtenář v [2] na stranách 195 – 198.

K lemmatu 4: Lemma 4 je v [2] uvedeno jako věta 6,2,1 na str. 152.

K lemmatu 5: Viz Poznámka 1 v [2] na str. 196.

K lemmatu 6: Viz Dodatek k Větě 6,2,1 v [2] na str. 152.

K lemmatu 7: Lemma 7 je v [2] uvedeno jako věta 6,2,9 na str. 158.

References

[1] R. B. Burckel: An Introduction to Classical Complex Analysis, Vol. 1. Birkhäuser Verlag Basel und Stuttgart 1979.

[2] I. Černý: Analýza v komplexním oboru. Academia Praha 1983.

[3] I. Černý: Foundations of Analysis in the Complex Domain. Academia Praha 1992.

[4] S. Saks – A. Zygmund: Analytic Functions. Warszawa – Wroclaw 1952.

Poděkování

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