ALGEBRA AND INFINITE COMBINATORICS

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§1 Test modules for projectivity and injectivity

A useful method for testing projectivity and injectivity of modules consists in evaluations of appropriate extension groups Ext. Of course, a module M is projective iff $Ext_R(M,N)=0$ for all modules $N \in Mod\text{-}R$. Similarly, N is injective iff $Ext_R(M,N)=0$ for all modules $M \in Mod\text{-}R$.

When testing the projectivity or injectivity, we need not check all the groups $Ext_R(M,N)$. In fact, the evaluation of a single group is enough. For each module M, there exist modules K and L such that M is projective (injective) iff $Ext_R(M,K)=0$ ($Ext_R(L,M)=0$). The problem is that the modules K and L depend on M, and they can be quite big provided M is such. We would like to get rid of this dependence, deciding the projectivity or injectivity of M by calculating a single Ext group using a fixed module N. This leads to the following basic definition:

Definition 1.1. *Let R be a ring and N be a module.*

(i) N is said to be a test module for projectivity (or a *p-test module*) provided for all $M \in Mod\text{-}R$, M is projective iff $Ext_R(M,N)=0$.
The class of all p-test modules is called the *p-test class* and denoted by $\mathcal{PT}$.

(ii) N is said to be a *test module for injectivity* (or an *i-test module*) provided for all $M \in Mod\text{-}R$, M is injective iff $Ext_R(N,M)=0$.
The class of all i-test modules is called the *i-test class* and denoted by $\mathcal{IT}$.

In this section, we deal with existence of p-test and i-test modules. We show that the i-test class is a proper class for an arbitrary ring. Further, the p-test class is a proper class for any right-perfect ring. The more difficult question of existence of p-test modules over non-right perfect rings will be studied in the next two sections, since the answer cannot be given using only algebraic methods.

First, we note that projectivity and injectivity of a given module can be tested by checking a single Ext group:

Lemma 1.2. *Let R be a ring.*
(i) Let $M \in Mod\text{-}R$ and

$$0 \to K \xrightarrow{\nu} P \to M \to 0$$

be a short exact sequence in $Mod\text{-}R$ such that P is projective. Then M is projective iff $Ext_R(M,K)=0$.
(ii) Let $N \in Mod\text{-}R$ and

$$0 \to N \to I \to L \to 0$$

be a short exact sequence in $Mod\text{-}R$ such that I is injective. Then N is injective iff $Ext_R(L,N)=0$.

Proof. (i) Using the definition of Ext by Hom groups of the given projective presentation of M, we get $Ext_R(M,K) \simeq Hom_R(K,K)/Im(Hom_R(\nu,K))$. Assume $Ext_R(M,K)=0$. Then $id_K = \pi\nu$, for some $\pi \in Hom_R(P,K)$. So $Ker(\pi)$ is a summand of P and $Ker(\pi) \simeq P/Im(\nu) \simeq M$.
(ii) Dual to (i). □

Using Baer's criterion for injectivity, it is easy to see that i-test modules exist over an arbitrary ring R:

Proposition 1.3. *Let R be a ring. Let $\mathcal{E}$ be the set of all proper essential right ideals of R. Put $M = \oplus \sum_{I \in \mathcal{E}} R/I$. Then M is an i-test module.*

Proof. Assume $Ext_R(M,N) = 0$. Let J be a left ideal of R and $\phi \in Hom_R(J,N)$. There exist $J \subseteq I \in \mathcal{E}$ and $\bar{\phi} \in Hom_R(I,N)$ such that $\bar{\phi} \restriction J = \phi$. By the premise, $Ext_R(R/I,N) = 0$. Since the sequence

$$0 \to Hom_R(R/I,N) \to Hom_R(R,N) \to Hom_R(I,N) \to Ext_R(R/I,N) = 0$$

is exact, there is some $\varphi \in Hom_R(R,N)$ such that $\varphi \restriction I = \bar{\phi}$. By the Baer's criterion, N is injective. □

Of course, any module possessing a summand isomorphic to an i-test module is likewise i-test. This implies

Corollary 1.4. *For any ring R, there is a proper class of i-test modules.*

Clearly, each (projective) module is i-test iff the ring R is semisimple. Denote by $\mathcal{P}$ the class of all projective modules. Then $\mathcal{IT} \subseteq Mod\text{-}R \setminus \mathcal{P}$ provided R is not semisimple. Though $\mathcal{IT}$ is a proper class, almost never does $\mathcal{IT} = Mod\text{-}R \setminus \mathcal{P}$. We shall consider this problem in detail in §4.

Proposition 1.5. *Let R be a right perfect ring. Denote by $\mathcal{M}$ the set of all maximal right ideals of R. Put $N = \oplus \sum_{I \in \mathcal{M}} R/I$. Then N is a p-test module.*

Proof. Assume $Ext_R(M,N) = 0$ and M is non-projective. Let

$$0 \to K \to P \to M \to 0$$

be a projective cover of M, i.e. a short exact sequence with P projective and K a superfluous submodule of P. By the premise, $K \neq 0$ and K has a maximal submodule, L. Then K/L is isomorphic to a summand of N, and $Ext_R(M,K/L) = 0$. Let $\pi \in Hom_R(K,K/L)$ be the projection. By the definition of Ext using Hom groups of the projective cover of M, there is a $\phi \in Hom_R(P,K/L)$ such that $\phi \restriction K = \pi$. Then $Ker(\phi)$ is a maximal submodule of P and $K \subseteq Rad(P) \subseteq Ker(\phi) \subset P$. Thus, $\pi = \phi \restriction K = 0$, a contradiction. □

Corollary 1.6. *Let R be a right perfect ring. Then there is a proper class of p-test modules.*

Clearly, each (injective) module is p-test iff the ring R is semisimple. Denote by $\mathcal{I}$ the class of all injective modules. Then $\mathcal{PT} \subseteq Mod\text{-}R \setminus \mathcal{I}$ provided R is not semisimple. Though $\mathcal{PT}$ is a proper class over any right perfect ring, almost never does $\mathcal{PT} = Mod\text{-}R \setminus \mathcal{I}$. Also this problem will be considered in detail in §4.

§2 The use of uniformization principles

In this and in the subsequent section, we show how methods and results of infinite combinatorics are used to answer the question of existence of p-test modules over non-right perfect rings. The answer turns out to be independent of ZFC.

In this section, we show that there is no p-test module over any non-right perfect ring, assuming Shelah's uniformization principle UP. Moreover, there exist non-right perfect rings over which there are no finitely generated p-test modules (in ZFC).

We start with several notions from infinite combinatorics:

Definition 2.1. *Let R be a non-right perfect ring. By Bass' Theorem P, there exist elements $a_i \in R, i < \aleph_0$, such that $(Ra_i \dots a_0; i < \aleph_0)$ is a strictly decreasing chain of principal left ideals of R. Let κ be an infinite cardinal and E be a subset of κ^+ such that $E \subseteq \{\alpha < \kappa^+; \mathrm{cf}(\alpha) = \aleph_0\}$. Let $(n_\nu; \nu \in E)$ be a ladder system* , i.e. for each $\nu \in E$, let $(n_\nu(i); i < \aleph_0)$ be a strictly increasing sequence of non-limit ordinals less that ν such that $\sup_{i<\aleph_0} n_\nu(i) = \nu$.
Let $(R_\alpha; \alpha < \kappa^+)$ be a system of free modules defined as follows: $R_\alpha = R$ provided $\alpha \in \kappa^+ \setminus E$, and $R_\alpha = R^{(\aleph_0)}$ provided $\alpha \in E$. For $\alpha \in \kappa^+ \setminus E$, denote by 1_α the canonical generator of R_α, and for $\alpha \in E$ let $\{1_{\alpha,i} \mid i < \aleph_0\}$ be the canonical basis of R_α. Note that by Bass' lemma, for every $\nu \in E$, the module

$$S_\nu = \sum_{i<\aleph_0} (1_{\nu,i} - 1_{\nu,i+1} \cdot a_i)R$$

is a free submodule of R_ν such that R_ν/S_ν is not projective. Put $P = \oplus \sum_{\alpha<\kappa^+} R_\alpha$, $Q = \sum_{\alpha \in E} Q_\alpha$, and $Q_\alpha = \sum_{i<\aleph_0} g_{\alpha i}R$ for all $\alpha \in E$, where $g_{\alpha i} = (1_{n_\alpha(i)} - 1_{\alpha,i} + 1_{\alpha,i+1} \cdot a_i) \in P$, for all $\alpha \in E$ and $i < \aleph_0$. Finally, put $M = P/Q \in Mod\text{-}R$.
Recall that E is a *stationary* subset of a cardinal λ provided E has a non-empty intersection with any closed and cofinal subset of λ.

Lemma 2.2. *If E is a stationary subset of κ^+, then $proj.dim(M) = 1$.*

Proof. Put $M_0 = 0$ and, for each $0 < \alpha < \kappa^+$, $M_\alpha = (\oplus \sum_{\beta<\alpha} R_\beta + Q)/Q$. Then $(M_\alpha; \alpha < \kappa^+)$ is a κ^+-filtration of M. Clearly, $Q = \oplus \sum_{\alpha \in E} Q_\alpha$, $Q_\alpha = \oplus \sum_{i<\aleph_0} g_{\alpha i}R$ for all $\alpha \in E$, and $Ann(g_{\alpha i}) = 0$ for all $\alpha \in E$ and $i < \aleph_0$. Hence, $proj.dim(M) \leq 1$. Proving indirectly, assume $proj.dim(M) = 0$, i.e. M is projective. By Kaplansky's structure theorem for projective modules, there exist modules $(P_\alpha; \alpha < \kappa^+)$ such that $\mathrm{gen}(P_\alpha) \leq \aleph_0$ for all $\alpha < \kappa^+$ and $M = \oplus \sum_{\alpha<\kappa^+} P_\alpha$. Put $N_0 = 0$ and, for each $0 < \alpha < \kappa^+, N_\alpha = \oplus \sum_{\beta<\alpha} P_\beta$. Clearly, $(N_\alpha; \alpha < \kappa^+)$ is a κ^+-filtration of M. Since the set $C = \{\alpha < \kappa^+; M_\alpha = N_\alpha\}$ is closed and cofinal in κ^+, Eklof's lemma implies there exists $\nu \in E \cap C$. Of course, $D = C \cap \{\alpha < \kappa^+; \nu < \alpha\}$ is also closed and cofinal in κ^+, whence there is some $\mu \in E \cap D$. Then $X = N_\mu/N_\nu$ is a projective module. On the other hand, put $Y = (\oplus \sum_{\nu<\alpha<\mu} R_\alpha + Q)/Q$. Then $X = M_\mu/M_\nu = M_{\nu+1}/M_\nu + (Y + M_\nu)/M_\nu$. By 2.1, $Y \cap M_{\nu+1} \subseteq M_\nu$, whence $M_{\nu+1}/M_\nu \simeq R_\nu/S_\nu$ is a non-projective summand of X, a contradiction. □

The following (meta-) lemma is proved by forcing in [Sh, §2] or [ESh, §2]:

Lemma 2.3. *Let κ be a cardinal such that* $\mathrm{cf}(\kappa) = \aleph_0$. *Consider the following assertion*
UP_κ: *"there exist a stationary subset E of κ^+ satisfying $E \subseteq \{\alpha < \kappa^+; \mathrm{cf}(\alpha) = \aleph_0\}$ and a ladder system $(n_\nu; \nu \in E)$ such that for each cardinal $\lambda < \kappa$ and each sequence $(h_\nu; \nu \in E)$ of mappings from $\aleph_0$ to λ there is a mapping $f : \kappa^+ \to \lambda$ such that $\forall\, \nu \in E\ \exists\, j < \aleph_0\ \forall\, j < i < \aleph_0 : f(n_\nu(i)) = h_\nu(i)$".*
Denote by UP the assertion "UP_κ *holds for every uncountable cardinal κ such that* $\mathrm{cf}(\kappa) = \aleph_0$ *". Then UP is consistent with ZFC + GCH.*

The principle UP_κ says that there are a stationary set $E \subseteq \kappa^+$ and a ladder system such that for all colourings of all ladders from the ladder system by $< \kappa$ colours there is a uniform colouring of the whole κ^+ which coincides with each particular colouring on almost all steps of the particular ladder.

Lemma 2.4. *Let κ be a cardinal such that* $\operatorname{cf}(\kappa) = \aleph_0$. *Assume* UP_κ *holds. Let* $M = P/Q$ *be the module corresponding to the* E *and* $(n_\nu(i); \nu \in E)$ *from* UP_κ *by* 2.1. *Then* $\operatorname{Ext}_R(M,N) = 0$ *for all* $N \in R\text{-}\mathrm{mod}$ *such that* $\operatorname{card}(N) < \kappa$.

Proof. We have $\operatorname{Ext}_R(M,N) \simeq \operatorname{Hom}_R(Q,N)/Im(\operatorname{Hom}_R(\tau,N))$, τ being the inclusion of Q into P. We are to prove that every $x \in \operatorname{Hom}_R(Q,N)$ is a restriction of some $y \in \operatorname{Hom}_R(P,N)$, i.e. $x = y\tau$. Take $x \in \operatorname{Hom}_R(Q,N)$. W.l.o.g., we can assume that the support of the module N is $\lambda = \operatorname{card}(N)$. Using the notation of 2.1, for each $\nu \in E$, we define $h_\nu : \aleph_0 \to \lambda$ by $h_\nu(i) = x(g_{\nu i})$ for all $i < \aleph_0$. By UP_κ, there exists $f : \kappa^+ \to \lambda$ such that $\forall\, \nu \in E \exists\, j_\nu < \aleph_0 \,\forall\, j_\nu < i < \aleph_0 : h_\nu(i) = f(n_\nu(i))$. Define $y \in \operatorname{Hom}_R(P,N)$ as follows: Take $\alpha < \kappa^+$.

If $\alpha = n_\nu(i)$ for some $\nu \in E$ and $j_\nu < i < \aleph_0$, put $y(1_\alpha) = f(\alpha)$;

If α does not satisfy (I) and $\alpha \notin E$, put $y(1_\alpha) = 0$;

If $\alpha \in E$, put $y(1_{\alpha,i}) = 0$ provided $i > j_\alpha$. For $0 \le i \le j_\alpha$, define $y(1_{\alpha,i})$ by induction on i (downwards): If there exist $\nu \in E$ and $k > j_\nu$ such that $n_\alpha(i) = n_\nu(k)$, put $y(1_{\alpha,i}) = f(n_\alpha(i)) - x(g_{\alpha i}) + y(1_{\alpha,i+1}) \cdot a_i$. If there are no $\nu \in E$ and $k > j_\nu$ such that $n_\alpha(i) = n_\nu(k)$, put $y(1_{\alpha,i}) = -x(g_{\alpha i}) + y(1_{\alpha,i+1}) \cdot a_i$.

It remains to prove that $x(g_{\alpha i}) = y(g_{\alpha i})$ for all $\alpha \in E$ and $i < \aleph_0$. Put $\beta = n_\alpha(i)$. Of course, $y(g_{\alpha i}) = y(1_\beta) - y(1_{\alpha,i}) + y(1_{\alpha,i+1}) \cdot a_i$. We distinguish the following three cases:

$i > j_\alpha$. Then $y(1_\beta) = f(\beta) = h_\alpha(i) = x(g_{\alpha i})$ and $y(1_{\alpha,i}) = y(1_{\alpha,i+1}) = 0$, whence $y(g_{\alpha i}) = x(g_{\alpha i})$;

$i \le j_\alpha$, but there exist $\nu \in E$ and $k > j_\nu$ such that $\beta = n_\nu(k)$. Then $y(1_\beta) = f(\beta)$ and $y(1_{\alpha,i}) = f(\beta) - x(g_{\alpha i}) + y(1_{\alpha,i+1}) \cdot a_i$, whence $y(g_{\alpha i}) = x(g_{\alpha i})$;

$i \le j_\alpha$ and there are no $\nu \in E$ and $k > j_\nu$ such that $\beta = n_\nu(k)$. Then $y(1_\beta) = 0$ and $y(1_{\alpha,i}) = -x(g_{\alpha i}) + y(1_{\alpha,i+1}) \cdot a_i$ whence $y(g_{\alpha i}) = x(g_{\alpha i})$, q.e.d. □

Theorem 2.5. *The assertion "There is no p-test module over any non-right perfect ring" is consistent with ZFC + GCH.*

Proof. By 2.3, we assume UP. Let N be a module. Let κ be a cardinal such that $\kappa > card(N)$. By 2.2 and 2.4, there is a non-projective module M such that $Ext_R(M,N) = 0$. Hence, N is not a p-test module. □

The following example shows (in ZFC) that there exist non-right perfect rings without finitely generated p-test modules:

Example 2.6. *Let R be a right self-injective non-right perfect ring (e.g. let R be the ring of all linear transformations of an infinite dimensional right linear space over a skew-field). Then no finitely generated module is a p-test module.*

Proof. Let a_i, $i < \aleph_0$ be as in 2.1. Let 1_i, $i < \aleph_0$ be the canonical basis of the free module $F = R^{(\aleph_0)}$ and let $G = \sum_{i<\aleph_0}(1_i - 1_{i+1} \cdot a_i)R \subseteq F$. Put $M = F/G$. By Bass' lemma, G is a free module, and M is not projective. If N is a finitely generated module, we have $N \simeq R^{(n)}/K$ for some $n < \aleph_0$ and $K \subseteq R^{(n)}$. Since the sequence $0 \to G \to F \to M \to 0$ is exact, we get $0 = Ext_R(G,K) \to Ext^2_R(M,K) \to Ext^2_R(F,K) = 0$, and $Ext^2_R(M,K) = 0$. Since the sequence $0 \to K \to R^{(n)} \to N \to 0$ is exact and R is right self-injective, we have $0 = Ext_R(M,R^{(n)}) \to Ext_R(M,N) \to Ext^2_R(M,K) = 0$, whence $Ext_R(M,N) = 0$. □

For right hereditary rings, the question of existence of p-test modules can be decided on free modules:

Proposition 2.7. *Let R be a ring and κ a cardinal. Then the following conditions are equivalent:*
(i) There exists a p-test module N such that $gen(N) \leq \kappa$ and $proj.dim(N) \leq 1$;
(ii) $R^{(\kappa)}$ is a p-test module.

Proof. The non-trivial part is (i) $\implies$ (ii) : Let M be a module such that $Ext_R(M, R^{(\kappa)}) = 0$. By the premise, there is an exact sequence $0 \to K \to R^{(\kappa)} \to N \to 0$, where K is projective. Then $0 = \mathrm{Ext}_R(M, R^{(\kappa)}) \to \mathrm{Ext}_R(M, N) \to \mathrm{Ext}^2_R(M, K) = 0$, whence $\mathrm{Ext}_R(M, N) = 0$, and M is projective by (i). □

Corollary 2.8. *Let R be a right hereditary ring. If there is a p-test module in Mod-R, then there is a cardinal κ such that each free module of rank $\geq \kappa$ is p-test.*

Proof. Take $\kappa = min\{gen(N); N \text{ is p-test }\}$ and apply 2.7. □

§3 Weak diamonds and the existence of p-test modules

The main purpose of this section is to prove consistency of existence of p-test modules for certain classes of non-right perfect rings. An essential tool for this is a combinatorial principle called generalized weak diamond (and denoted by Ψ). Since Ψ is a consequence of the axiom of constructibility, all consequences of Ψ are consistent with ZFC + GCH. Our proof is in three steps:
Step I: by purely algebraic means, the existence of modules testing projectivity of modules of "small" size is obtained;
Step II: using Ψ, the testing is extended to modules of regular cardinality;
Step III: Shelah's Compactness Theorem is applied to cover the singular cardinality case.
Note that the proof requires the generalized weak diamond only in Step II, the other steps being proved in ZFC.

In this way, the existence of p-test modules is achieved for all right hereditary non-right perfect rings. Further results are obtained in the particular cases when
(1) R is a Dedekind domain with $card(R) \leq \aleph_1$ such that R is not a complete discrete valuation ring; and
(2) R is a simple von Neumann regular ring with $card(R) \leq \aleph_1$ such that R has countable dimension over its center.
In the case 1), p-test modules include all non-zero free modules. In the case 2), all non-zero countably generated modules are p-test. Hence, also all non-zero free modules, and semisimple modules, are p-test in the case 2).

Step I for the Dedekind domains is a well-known generalization of the classical result of Stein for $\mathbb{Z}$. The generalization is due to Nunke ([Nu, §8]):

Proposition 3.1. *Let R be a Dedekind domain which is not a complete discrete valuation ring. Let F be a non-zero free module, and M be a module of countable rank. Then M is projective iff $Ext_R(M, F) = 0$.*

We turn to Step I for the von Neumann regular rings:

Lemma 3.2. *Let R be a von Neumann regular ring such that $dim_K(R) \leq \aleph_0$, K being the center of R. Then each left (right) ideal of R is countably generated. Hence R is (left and right) hereditary. In particular, any countable von Neumann regular ring is hereditary.*

Proof. Let I be a right ideal of R. Then $dim_K(I) \leq \aleph_0$. Let $B = \{b_n; n < \kappa\}$, $\kappa \leq \aleph_0$, be a left K-basis of I. Then $I = \sum_{n<\kappa} b_n R$, and I is countably generated. Since R is regular, I is projective, and R is right hereditary. The left-hand version follows similarly. □

Proposition 3.3. *Let R be a simple von Neumann regular ring. Denote by K the center of R. Assume $dim_K(R) \leq \aleph_0$. Let N be a non-zero countably generated module. Let M be any countably generated module. Then M is projective iff $Ext_R(M, N) = 0$.*

Proof. Let $Ext_R(M, N) = 0$. Let M' be any finitely generated submodule of M. We prove that M' is projective: We have $M' \simeq R^{(m)}/I$ for some $0 < m < \aleph_0$ and $I \in Mod\text{-}R$. Proving indirectly, assume $gen(I) \geq \aleph_0$. Then the regularity of R implies I is a direct sum of cyclic modules, $I = \oplus \sum_{\alpha<\kappa} x_\alpha R$. Since $Ext_R(M, N) = 0$, 3.2 implies $Ext_R(M', N) = 0$. Then also $Ext_R(R^{(m)}/I', N) = 0$, where $I' = \oplus \sum_{\alpha \in C} x_\alpha R$, for a countably infinite subset $C \subseteq \kappa$. Since each $x_\alpha R$ is cyclic and projective, there exist idempotents $0 \neq e_\alpha$, $\alpha \in C$, such that $x_\alpha R \simeq e_\alpha R$ for all $\alpha \in C$. Since R is a simple ring, we have $N e_\alpha \neq 0$ for all $\alpha \in C$. In particular, $dim_K(Hom_R(I', N)) = dim_K(\prod_{\alpha \in C} N e_\alpha) \geq dim_K(K^{\aleph_0}) > \aleph_0$, while $dim_K(Hom_R(R^{(m)}, N)) = dim_K(N^{(m)}) \leq \aleph_0$. This contradicts the exactness of the sequence

$$0 \to Hom_R(R^{(m)}/I', N) \to Hom_R(R^{(m)}, N) \to Hom_R(I', N) \to 0.$$

Hence, $gen(I) < \aleph_0$, I is a summand of $R^{(m)}$ (as R is regular), and M' is projective. Thus, M is $\aleph_0$-projective, and the assertion is true provided M is finitely generated. If $gen(M) = \aleph_0$, we use the following

Lemma 3.4. *Let R be a right hereditary von Neumann regular ring. Let M be an $\aleph_0$-projective module with $gen(M) = \aleph_0$. Then M is projective.*

Proof. We have $M = R^{(\aleph_0)}/I$ for some $I \subseteq R^{(\aleph_0)}$. Put $M_n = (R^{(n)} + I)/I$ and $I_n = R^{(n)} \cap I$, $n < \aleph_0$. Then M is a union of the non-decreasing chain $(M_n; n < \aleph_0)$. By the premise, $M_n \simeq R^{(n)}/I_n$ is projective, and I_n is finitely generated. Therefore, we can define two sets, $(A_n; 0 < n < \aleph_0)$, and $(B_n; 0 < n < \aleph_0)$, of finitely generated submodules of $R^{(\aleph_0)}$ by

$$B_n \oplus (I_n + R^{(n-1)}) = R^{(n)} \quad \text{and} \quad I_{n+1} = I_n \oplus A_n,$$

for each $0 < n < \aleph_0$. Then $R^{(\aleph_0)} = I_1 \oplus (\oplus \sum_{0<n<\aleph_0} A_n) \oplus (\oplus \sum_{0<n<\aleph_0} B_n)$. Now, $I = \cup_{0<n<\aleph_0} I_n$, $I = I_1 \oplus (\oplus \sum_{0<n<\aleph_0} A_n)$, and $M \simeq \oplus \sum_{0<n<\aleph_0} B_n$ is projective. □

For Step II, we start with combinatorial principles that follow from the axiom of constructibility:

Definition 3.5. *Let κ be a regular uncountable cardinal and E be a stationary subset of κ. Denote by $\diamondsuit_\kappa(E)$ the Jensen's diamond*(for κ and E), i.e. the assertion

" Let A be any set of cardinality κ and $(A_\alpha; \alpha < \kappa)$ a κ-filtration of A. Then there is a system $\{S_\alpha; \alpha < \kappa\}$ such that $S_\alpha \subseteq A_\alpha$ for all $\alpha < \kappa$, and the set $\{\alpha \in E; X \cap A_\alpha = S_\alpha\}$ is stationary in κ, for every $X \subseteq A$."

Denote by $\Phi_\kappa(E)$ the *weak diamond*(for κ and E), i.e. the assertion

" Let A be any set of cardinality κ and $(A_\alpha; \alpha < \kappa)$ a κ-filtration of A. For each $\alpha \in E$, let $P_\alpha : Exp(A_\alpha) \to \{0,1\}$ be given. Then there is $\phi : E \to \{0,1\}$ such that the set $\{\alpha \in E; P_\alpha(X \cap A_\alpha) = \phi(\alpha)\}$ is stationary in κ, for every $X \subseteq A$."

Denote by $\Psi_\kappa(E)$ the assertion

" Let A be any set of cardinality κ and $(A_\alpha; \alpha < \kappa)$ a κ-filtration of A. For each $\alpha \in E$, let $2 \leq p_\alpha < \aleph_0$ and let $P_\alpha : Exp(A_\alpha) \to p_\alpha$ be given. Then there is $\psi : E \to \aleph_0$ such that $\psi(\alpha) \in p_\alpha$ for all $\alpha \in E$, and the set $\{\alpha \in E; P_\alpha(X \cap A_\alpha) = \psi(\alpha)\}$ is stationary in κ, for every $X \subseteq A$."

Finally, denote by Ψ the *generalized weak diamond*, i.e. the assertion:

" $\Psi_\kappa(E)$ holds for each regular uncountable cardinal κ and each stationary subset $E \subseteq \kappa$ ".

Note that $\Psi_\kappa(E)$ says that given colourings of all subsets of all members of a filtration of A by a varying finite number of colours, there is a colour estimate function ψ which works well for a large number of indices. This principle is less well-known, but it will be very useful to us. We notice its position between the better known diamond principles:

Lemma 3.6. *(i) Let κ be a regular uncountable cardinal and E be a stationary subset of κ. Then $\diamondsuit_\kappa(E) \implies \Psi_\kappa(E) \implies \Phi_\kappa(E)$.*
(ii) Ψ is consistent with ZFC + GCH.

Proof. (i) This is clear, taking $\psi(\alpha) = P_\alpha(S_\alpha)$ for all $\alpha \in E$ for the first implication, and taking $p_\alpha = 2$ for all $\alpha < \kappa$ for the second.

(ii) Assume the axiom of constructibility. Then, by a well-known result of Jensen, $\diamondsuit_\kappa(E)$ holds for each regular uncountable cardinal κ and each stationary subset E of κ. Note that the Jensen's diamond for $\kappa = \lambda^+$ and $E = \lambda^+$ implies $2^\lambda = \lambda^+$. Hence, GCH holds, and (i) implies that the assertion holds true. $\square$

In general, by [Sh, Ch.XIV], none of the implications from 3.6(i) can be reversed.

Lemma 3.7. *Let κ be a regular uncountable cardinal and E a stationary subset of κ. Assume $\Psi_\kappa(E)$. Let R be a ring with $card(R) \leq \kappa$. Let N be a module such that $card(I(N)) \leq \kappa$. Let M be a κ-projective module such that $gen(M) = \kappa$ and there is a κ-filtration $(C_\alpha; \alpha < \kappa)$ of M such that $E = \{\alpha < \kappa; Ext_R(C_{\alpha+1}/C_\alpha, N) \neq 0\}$. Then $Ext_R(M,N) \neq 0$.*

Proof. First, we take a κ-filtration $(D_\alpha; \alpha < \kappa)$ of the set κ and elements $m_\alpha \in M$, $\alpha < \kappa$, such that $C_\alpha = \sum_{\beta \in D_\alpha} m_\beta R$, for all $\alpha < \kappa$. Let $(B_\alpha; \alpha < \kappa)$ be a κ-filtration of the $\mathbb{Z}$-module $I = I(N)$. Denote by ν the inclusion of N into I, by π the projection of I onto I/N, and by ν_α the inclusion of C_α into $C_{\alpha+1}$, for all $\alpha < \kappa$.

Take $\alpha \in E$. Let $X_\alpha = Hom_R(C_\alpha, N)$ and $Y_\alpha = Im(Hom(\nu_\alpha, N))$. By the premise, there is some $f_\alpha \in X_\alpha \setminus Y_\alpha$. Denote by o_α the order of $f_\alpha + Y_\alpha$ in the group $X_\alpha/Y_\alpha = Ext_R(C_{\alpha+1}/C_\alpha, N)$.

We are going to use the principle $\Psi_\kappa(E)$ in the following setting: $A = \kappa \times I$ and $A_\alpha = D_\alpha \times B_\alpha$, $\alpha < \kappa$. Let $\alpha \in E$. If $o_\alpha = \aleph_0$, we put $p_\alpha = 2$. If $o_\alpha < \aleph_0$, we define $p_\alpha = o_\alpha$. In order to define the colourings P_α, $\alpha \in E$, we equip the set of all mappings from D_α to B_α with an equivalence relation $\sim_\alpha$: we put $u \sim_\alpha v$ iff there are $n \in \mathbb{Z}$ and $y \in Y_\alpha$ such that $v = u + nf_\alpha \restriction D_\alpha + y \restriction D_\alpha$. Note that the number n is unique (unique modulo p_α) provided $o_\alpha = \aleph_0$ ($o_\alpha < \aleph_0$). Now, for each $\alpha \in E$,

we take a colouring $P_\alpha : Exp(A_\alpha) \to p_\alpha$ such that $P_\alpha(u) = P_\alpha(v)$ iff the number n given by the pair (u, v) is divisible by p_α.
Let $\psi : E \to \aleph_0$ be the mapping corresponding to this setting by $\Psi_\kappa(E)$. In order to prove that $Ext_R(M,N) \neq 0$, we shall construct $g \in Hom_R(M, I/N) \setminus Im(Hom_R(M,\pi))$. By induction on $\alpha < \kappa$, we define $g_\alpha \in Hom_R(C_\alpha, I/N)$ so that $g_{\alpha+1} \restriction C_\alpha = g_\alpha$ for each $\alpha < \kappa$, and $g_\alpha = \cup_{\beta<\alpha} g_\beta$ for all limit $\alpha < \kappa$.
Put $g_0 = 0$. Assume g_α is defined for an ordinal $\alpha < \kappa$. We distinguish the following two cases:
(I) $\alpha \in E$ and there exists $f \in Hom_R(C_{\alpha+1}, I)$ such that $Im(f\nu_\alpha \restriction D_\alpha) \subseteq B_\alpha$, $P_\alpha(f\nu_\alpha \restriction D_\alpha) = \psi(\alpha)$, and $g_\alpha = \pi f \nu_\alpha$.
(II) = not (I).
In the case (I), take an f satisfying the conditions of (I). The injectivity of I yields the existence of $h_\alpha \in Hom_R(C_{\alpha+1}, I)$ such that $h_\alpha\nu_\alpha = f\nu_\alpha - f_\alpha$. Put $g_{\alpha+1} = \pi h_\alpha$. Then $g_{\alpha+1}\nu_\alpha = \pi f\nu_\alpha - \pi f_\alpha = g_\alpha$.
In the case (II), the projectivity of C_α yields the existence of $h_\alpha \in Hom_R(C_\alpha, I)$ such that $g_\alpha = \pi h_\alpha$. The injectivity of I gives some $h_{\alpha+1} \in Hom_R(C_{\alpha+1}, I)$ such that $h_\alpha = h_{\alpha+1}\nu_\alpha$. Put $g_{\alpha+1} = \pi h_{\alpha+1}$. Then $g_{\alpha+1} \restriction C_\alpha = g_\alpha$.
Finally, put $g = \cup_{\alpha<\kappa} g_\alpha$. Then $g \in Hom_R(M, I/N)$. Proving indirectly, suppose there is $h' \in Hom_R(M, I)$ such that $g = \pi h'$. Note that the set $\{\alpha < \kappa; Im(h' \restriction D_\alpha) \subseteq B_\alpha\}$ is closed and cofinal in κ. Put $X = \cup_{\alpha<\kappa}(h' \restriction D_\alpha)$. By the premise, there is an $\alpha \in E$ such that $g \restriction C_\alpha = \pi h\nu_\alpha$, $P_\alpha(h\nu_\alpha \restriction D_\alpha) = P_\alpha(X \cap A_\alpha) = \psi(\alpha)$, and $Im(h\nu_\alpha \restriction D_\alpha) \subseteq B_\alpha$, where $h = h' \restriction C_{\alpha+1}$. Hence, the case (I) occurs, and $\pi(h_\alpha - h) = 0$. Then $y_\alpha = (h_\alpha - h)\nu_\alpha \in Y_\alpha$. Moreover, $f\nu_\alpha = h\nu_\alpha + f_\alpha + y_\alpha$, whence $\psi(\alpha) = P_\alpha(f\nu_\alpha \restriction D_\alpha) = P_\alpha(h\nu_\alpha \restriction D_\alpha + f_\alpha \restriction D_\alpha + y_\alpha \restriction D_\alpha) \neq P_\alpha(h\nu_\alpha \restriction D_\alpha)$, a contradiction. Thus $g \notin Im(Hom_R(M, \pi))$. □

Lemma 3.8. *Let κ be a regular uncountable cardinal. Assume $\Psi_\kappa(E)$ holds for all stationary subsets of κ. Let R be a ring with $card(R) \leq \kappa$. Let N be a module such that $card(I(N)) \leq \kappa$. Let M be a κ-projective module such that $gen(M) = \kappa$. Then the following conditions are equivalent:*
(i) $Ext_R(M,N) = 0$;
(ii) There is a κ-filtration $(C_\alpha; \alpha < \kappa)$ of M such that $Ext_R(C_{\alpha+1}/C_\alpha, N) = 0$ for all $\alpha < \kappa$.
Moreover, the implication (ii) $\implies$ (i) holds in ZFC.

Proof. (i) $\implies$ (ii): Since $gen(M) = \kappa$, there is a κ-filtration, $(D_\alpha; \alpha < \kappa)$, of the module M. By induction, we define a mapping $\eta : \kappa \to \kappa$ as follows. First, $\eta(0) = 0$. If $\eta(\alpha)$ is defined, then either $Ext_R(D_\beta/D_{\eta(\alpha)}, N) = 0$ for all $\beta \geq \eta(\alpha)$ and we put $\eta(\alpha+1) = \eta(\alpha)+1$, or there is a smallest $\eta(\alpha) < \beta < \kappa$ such that $Ext_R(D_\beta/D_{\eta(\alpha)}, N) \neq 0$ and we put $\eta(\alpha+1) = \beta$. For α limit, we put $\eta(\alpha) = sup_{\beta<\alpha}\eta(\beta)$. Then $(D_{\eta(\alpha)}; \alpha < \kappa)$ is a κ-filtration of the module M. Let $E = \{\alpha < \kappa; Ext_R(D_{\eta(\alpha+1)}/D_{\eta(\alpha)}, N) \neq 0\}$. By 3.7, E is not stationary in κ. Let C be a closed and cofinal subset of κ with $E \cap C = \emptyset$ and $0 \in C$. Let $\theta : \kappa \to C$ be a strictly increasing continuous mapping of κ onto C. For each $\alpha < \kappa$, put $C_\alpha = D_{\eta\theta(\alpha)}$. Then $(C_\alpha; \alpha < \kappa)$ is a κ-filtration of the module M satisfying (ii).
(ii) $\implies$ (i): We prove in ZFC. Denote by π the projection of $I = I(N)$ onto I/N. For $\alpha < \kappa$, let ν_α be the inclusion of C_α into $C_{\alpha+1}$. By induction on $\alpha < \kappa$ we define mappings $\varphi_\alpha : Hom_R(C_\alpha, I/N) \to Hom_R(C_\alpha, I)$ such that for each $f \in Hom_R(C_\alpha, I/N)$, $\pi\varphi_\alpha(f) = f$, and $\varphi_\alpha(f\nu_\alpha) = \varphi_\beta(f)\nu_\alpha$ provided $\beta = \alpha + 1$. For $\alpha = 0$ put $\varphi_\alpha = 0$. Let $0 < \alpha < \kappa$ and $f \in Hom_R(C_{\alpha+1}, I/N)$. Since $C_{\alpha+1}$

is projective, there is $g \in Hom_R(C_{\alpha+1}, I)$ with $f = \pi g$. As $g\nu_\alpha - \varphi_\alpha(f\nu_\alpha) \in Hom_R(C_\alpha, N)$ and $Ext_R(C_{\alpha+1}/C_\alpha, N) = 0$, there is $h \in Hom_R(C_{\alpha+1}, N)$ such that $h\nu_\alpha = g\nu_\alpha - \varphi_\alpha(f\nu_\alpha)$. We put $\varphi_{\alpha+1}(f) = g - h$. If α is limit, define $\varphi_\alpha = \cup_{\beta<\alpha}\varphi_\beta$. Then $f = \pi\varphi_\alpha(f)$, for all $f \in Hom_R(C_\alpha, I/N)$. Finally, put $\varphi = \cup_{\alpha<\kappa}\varphi_\alpha$. Then $f = \pi\varphi(f)$, for all $f \in Hom_R(C_\alpha, I/N)$, and $Ext_R(M, N) = 0$. □

Definition 3.9. *Let M be a module and $\lambda \geq \aleph_0$. Assume that for some submodules of M, sets called "bases" are given. If N is a submodule of M such that N has at least one "basis", we say that N is "free". We introduce the following axioms:*

(Ax I) If N is a "free" submodule of M and $\mathcal{F}$ is a "basis" of N, then $\mathcal{F}$ is a set of submodules of N, $\mathcal{F}$ is closed under unions of chains, and for each subset $A \subseteq N$ there is some $F \in \mathcal{F}$ such that $A \subseteq F$ and $card(F) \leq card(A) + \lambda$.

(Ax II) If N is a "free" submodule of M, $\mathcal{F}$ is a "basis" of N and $C \in \mathcal{F}$, then $\mathcal{F} \restriction C = \{D \in \mathcal{F}; D \subseteq C\}$ is a "basis" of C.

(Ax III) If N is a "free" submodule of M, C is an element of a "basis" of N, and C has a "basis" $\mathcal{G}$, then N has a "basis" $\mathcal{F}$ such that $\mathcal{G} = \mathcal{F} \restriction C$.

(Ax IV) Suppose $(N_\alpha; \alpha < \lambda)$ is a smooth chain of "free" submodules of M, for each $\alpha < \lambda$ a "basis" $\mathcal{F}_\alpha$ of N_α is given so that $\alpha < \beta < \lambda$ implies $\mathcal{F}_\alpha = \mathcal{F}_\beta \restriction N_\alpha$. Then $\cup_{\alpha<\lambda}N_\alpha$ has a "basis" consisting of all sets of the form $\cup_{\alpha<\lambda}C_\alpha$, where $(C_\alpha; \alpha < \lambda)$ is a chain of submodules of M, and $C_\alpha \in \mathcal{F}_\alpha$ for all $\alpha < \lambda$.

Now, we formulate the version of Shelah's Compactness Theorem that we shall need for Step III. Its proof, using game theoretic arguments, appears e.g. in [Ho, §4] or [EM, Ch.IV]:

Theorem 3.10. *Let R be a ring and M be a module such that $card(M) = \kappa$ is a singular cardinal. Let λ be an infinite cardinal such that $card(R) \leq \lambda < \kappa$. Assume the axioms (Ax I) - (Ax IV) from 3.8 hold, and every submodule of M of cardinality $< \kappa$ is "free". Then M is "free".*

Corollary 3.11. *Let R be a ring and M be a module such that $card(M) = \kappa$ is a singular cardinal. Assume that $card(R) < \kappa$ and M is κ-projective. Then M is projective.*

Proof. We shall say that a submodule N of M is "free" provided there are a cardinal μ and countably generated projective modules $P_\alpha \subseteq N$, $\alpha < \mu$, such that $N = \oplus\sum_{\alpha<\mu} P_\alpha$. Then the set $\mathcal{F} = \{C; \exists A \subseteq \mu : C = \oplus\sum_{\alpha\in A} P_\alpha\}$ is called a "basis" of the module N. Put $\lambda = card(R) \times \aleph_0$. Then each countably generated projective module has cardinality $\leq \lambda$. It is easy to see that the notions in quotes satisfy axioms (Ax I) - (Ax IV) of 3.9. On the other hand, Kaplansky's structure theorem for projective modules implies that a module is "free" iff it is projective. Hence, the premise and 3.10 imply that M is projective. □

Combining Steps II and III, we obtain

Theorem 3.12. *Assume Ψ. Let R be a right hereditary ring with $card(R) \leq \aleph_1$. Let N be a module such that $card(I(N)) \leq \aleph_1$. Assume that $Ext_R(M, N) = 0$ implies M is projective, for every countably generated module M. Then N is a p-test module.*

Proof. By induction on $gen(M) = \kappa$, we prove that M is projective whenever M is a module such that $Ext_R(M, N) = 0$. If $\kappa \leq \aleph_0$, the assertion holds by the premise. Let κ be a regular uncountable cardinal. Since R is right hereditary, 3.8 implies

M has a κ-filtration $(C_\alpha; \alpha < \kappa)$ such that $Ext_R(C_{\alpha+1}/C_\alpha, N) = 0$ for all $\alpha < \kappa$. By the induction premise, all the modules $C_{\alpha+1}/C_\alpha$, $\alpha < \kappa$, are projective, whence $M = \cup_{\alpha<\kappa} C_\alpha$ is projective.
Let κ be singular. Since R is right hereditary, M is κ-projective, by the induction premise. Now, 3.11 terminates the proof. □

For right hereditary rings, 2.8 shows that the existence of p-test modules implies the existence of free p-test modules. This occurs under Ψ:

Theorem 3.13. *Assume Ψ. Let R be a right hereditary ring.*
(i) Let $\kappa = card(R) \times \aleph_0$. Then each free module of rank $\geq 2^\kappa$ is p-test.
(ii) Let $\lambda \geq \aleph_0$ be a cardinal such that $card(I(R^{(\lambda)})) \leq \lambda$. Then each free module of rank $\geq \lambda$ is p-test.

Proof. (i) First, we prove that $card(I(R^{(2^\kappa)})) \leq 2^\kappa$. Put $M = R^{(2^\kappa)}$. Then there is a chain of modules

$$M \simeq Hom_R(R, M) \subseteq Hom_{\mathbb{Z}}(R, M) \subseteq Hom_{\mathbb{Z}}(R, D) = I,$$

where D is the divisible hull of the (right) $\mathbb{Z}$-module M. Since R is a flat left R-module, I is injective. Since $card(D) = card(M) = 2^\kappa$, we infer that $card(I) \leq (2^\kappa)^\kappa = 2^\kappa$. This proves $card(I(R^{(2^\kappa)})) \leq 2^\kappa$. Now, it suffices to show that
(ii) $R^{(\lambda)}$ is p-test provided λ is an infinite cardinal and $card(I(R^{(\lambda)})) \leq \lambda$. Assume that M is a $\leq \lambda$ generated module such that $Ext_R(M, R^{(\lambda)}) = 0$. Let K be a submodule of $R^{(\lambda)}$ such that $M \simeq R^{(\lambda)}/K$. By the premise, $gen(K) \leq \lambda$. Since R is right hereditary, we infer that $Ext_R(M, K) = 0$, and M is projective by 1.2(i). Now, starting from λ, and using 3.8 and 3.11 for induction in regular and singular cardinals, respectively, we obtain the claim. □

Stronger results hold true for the particular cases of Dedekind domains and of von Neumann regular rings:

Theorem 3.14. *Assume Ψ. Let R be a Dedekind domain such that R is not a complete discrete valuation ring, and $card(R) \leq \aleph_1$. Then any non-zero free module is p-test.*

Proof. It suffices to prove that R is p-test. Denote by K the quotient field of R. Then K is an injective module, $R \subseteq K$, and $card(K) = card(R) \leq \aleph_1$. Hence, 3.1 and 3.12 show that R is a p-test module. □

Theorem 3.15. *Assume Ψ. Let R be a right hereditary von Neumann regular ring such that $card(I(R^{(\aleph_0)})) \leq \aleph_1$. Then each free module of rank $\geq \aleph_0$ is p-test.*

Proof. By 3.4 and 3.12, we have to prove that $Ext_R(M, R^{(\aleph_0)}) = 0$ implies M is projective for each finitely generated module M. Let $M \simeq R^{(n)}/K$. Since R is regular, there are a cardinal κ and elements $0 \neq x_\alpha \in K$, $\alpha < \kappa$, such that $K = \oplus \sum_{\alpha<\kappa} x_\alpha R$.
Proving indirectly, assume $\kappa \geq \aleph_0$. Take a system of pairwise disjoint sets $A_k, k < \aleph_0$, such that $card(A_k) = n$ for each $k < \aleph_0$, and $\aleph_0 = \cup_{k<\aleph_0} A_k$. For each $k < \aleph_0$, we identify $R^{(n)}$ with $R^{(A_k)}$ via an R-isomorphism ν_k. Define $f \in Hom_R(K, R^{(\aleph_0)})$ by $f(x_\alpha) = \nu_\alpha(x_\alpha)$ provided $\alpha < \aleph_0$, and by $f(x_\alpha) = 0$ otherwise. Let $g \in Hom_R(R, R^{(\aleph_0)})$. Then $Im(g) \subseteq R^{(m)}$ for some $m < \aleph_0$, and $g \restriction K \neq f$. Then $Ext_R(M, R^{(\aleph_0)}) \neq 0$, a contradiction.
Hence, κ is finite, and M is projective (as R is regular). □

Lemma 3.16. *Let R be a von Neumann regular ring. Let N be a module and λ be a cardinal of cofinality ω. Denote by π_ν the ν-th canonical projection of M^λ to M, $\nu < \lambda$. Let $\{\lambda_k; k < \aleph_0\}$ be a cofinal subset of λ. For each $k < \aleph_0$ put $M_k = \{m \in M^\lambda; \pi_\nu(m) = 0 \,\forall \nu \geq \lambda_k\}$, and $M_\lambda = \cup_{k<\aleph_0} M_k$. Let J be a countably generated right ideal of R. Then $Ext_R(R/J, M^\lambda/M_\lambda) = 0$.*

Proof. If J is finitely generated, then R/J is projective (as R is regular), and the assertion is clear. If $gen(J) = \aleph_0$, the regularity of R implies there is a set, $\{e_n; n < \aleph_0\}$, of orthogonal idempotents of R such that $J = \oplus \sum_{n<\aleph_0} e_n R$. We have to extend each $\varphi \in Hom_R(J, M^\lambda/M_\lambda)$ to some $\phi \in Hom_R(R, M^\lambda/M_\lambda)$. We have $\varphi(e_n) = (x^n_\alpha.e_n; \alpha < \lambda) + N_\lambda$ for some $x^n_\alpha \in N$, $n < \aleph_0$, $\alpha < \lambda$. For $\alpha < \lambda_0$, put $y_\alpha = 0$. If $\lambda_k \leq \alpha < \lambda_{k+1}$, put $y_\alpha = \sum_{n\leq k} x^n_\alpha.e_n$. Define $\phi \in Hom_R(R, M^\lambda/M_\lambda)$ by $\phi(1) = (y_\alpha; \alpha < \lambda)$. Since the idempotents $e_n, n < \aleph_0$ are orthogonal, we have $\phi \restriction J = \varphi$. □

Proposition 3.17. *Assume Ψ and CH. Let R be a von Neumann regular ring such that each right ideal is countably generated and $card(R) \leq \aleph_1$. Let N be a module such that $gen(N) \leq \aleph_1$. Assume that $Ext_R(M, N) = 0$ implies M is projective, for every finitely generated module M. Then N is a p-test module.*

Proof. First, we use 3.16 for $\lambda = \aleph_0$ and $\lambda_n = n$, $n < \aleph_0$. Then $N_\lambda = N^{(\aleph_0)}$. Put $I = N^{\aleph_0}/N^{(\aleph_0)}$. By Baer's criterion and 3.16, I is injective. Denote by ν the mapping assigning each $x \in N$ the coset of the constant sequence $(x; k < \aleph_0)$. Then ν is an embedding of N into I. Note that $card(I) \leq \aleph_1^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = \aleph_1$. So the injective hull $I(N)$ of N has cardinality at most $\aleph_1$. Finally, the regularity of R and 3.4 show that we can apply 3.12. □

Corollary 3.18. *Assume Ψ and CH. Let R be a von Neumann regular ring such that each right ideal is countably generated and $card(R) \leq \aleph_1$. Then each free module of rank $\geq \aleph_0$ is p-test.*

Proof. Since $card(R) \leq \aleph_1$, also $card(I(R^{(\aleph_0)})) \leq \aleph_1$, and we use 3.15. □

The main application of 3.17 is to the case when R is simple and of countable dimension over its center:

Corollary 3.19. *Assume Ψ and CH. Let R be a simple von Neumann regular ring such that $card(R) \leq \aleph_1$ and $dim_K(R) \leq \aleph_0$, K being the center of R. Let N be a non-zero module such that N is either (i) countably generated, or (ii) projective, or (iii) semisimple. Then N is a p-test module.*

Proof. Part (i) follows from 3.2, 3.3 and 3.17. Parts (ii) and (iii) follow from the fact that the respective modules possess non-zero cyclic summands. □

There is no analogue (in ZFC) to 3.13-9 for arbitrary non-right perfect rings:

Example 3.20. *Let κ be an uncountable cardinal, K be a skew-field and M be a right linear K-space of dimension κ over K. Let R be the ring of all linear transformations of M. Then no module N with $proj.dim(N) \leq 1$ is p-test. In particular, no free module is p-test.*

Proof. Let $\{b_\alpha; \alpha < \kappa\}$ be a basis of M. Define a system of idempotents $\{e_\alpha; \alpha < \aleph_1\}$ of R as follows: $e_\alpha(b_\beta) = b_\beta$ provided $\beta \leq \alpha$, and $e_\alpha(b_\beta) = 0$ otherwise. Define a chain of right ideals of R by $I_0 = 0$, $I_{\alpha+1} = e_\alpha R$, $\alpha < \aleph_1$, and by

$I_\alpha = \cup_{\beta<\alpha} I_\beta$ provided α is a limit ordinal $< \aleph_1$. Then $(I_\alpha; \alpha < \aleph_1)$ is an $\aleph_1$-filtration of the right ideal $I = \cup_{\alpha<\aleph_1} I_\alpha$. Since R is von Neumann regular, $I_{\alpha+1}$ is a summand in I_β for all $\alpha < \beta < \aleph_1$. Hence, the set $A = \{\alpha < \aleph_1; \{\alpha < \beta < \aleph_1; I_\beta/I_\alpha$ is not projective $\}$ is stationary in $\aleph_1\}$ contains all limit ordinals $< \aleph_1$, and A is a stationary subset of $\aleph_1$. By Eklof's lemma and Kaplansky's structure theorem for projective modules, I is a non-projective right ideal of R.
We prove that $Ext_R(R/I, R^{(\lambda)}) = 0$ for each cardinal λ. Let $\phi \in Hom_R(I, R^{(\lambda)})$. For each $\alpha < \aleph_1$, denote by F_α the smallest finite subset of λ such that $\phi(e_\alpha) \in R^{(F_\alpha)}$. Then $(F_\alpha; \alpha < \aleph_1)$ is a non-decreasing chain of finite subsets of λ. Since $cf(\aleph_1) > \omega$, we infer that $F = \cup_{\alpha<\aleph_1} F_\alpha$ is a finite set, and $Im(\phi) \subseteq R^{(F)}$. Since R is right self-injective, there is some $\varphi \in Hom_R(R, R^{(F)})$ such that $\varphi \restriction I = \phi$. Then $Ext_R(R/I, R^{(\lambda)}) = 0$, and $R^{(\lambda)}$ is not p-test.
Finally, by 2.7, no module N with $proj.dim(N) \leq 1$ is p-test. □

§4 Rings possessing many test modules

By §1, we know that there is a proper class of i-test modules over any ring R. This suggests the question of how close can $\mathcal{IT}$ be to Mod-R. Obviously, $\mathcal{IT} = Mod$-R if and only if R is semisimple. If R is not semisimple, then no projective module is i-test. Thus, investigating the possible size of $\mathcal{IT}$, we start with the question whether $\mathcal{IT}$ can contain all "small" non-projective modules.

Since similar observations apply to the dual case of the class $\mathcal{PT}$, we arrive at the following

Definition 4.1. Let R be a non-semisimple ring and κ be a cardinal. Then $\mathcal{IT}$ ($\mathcal{PT}$) is said to be *κ-saturated* provided it contains all non-projective (non-injective) modules M such that $gen(M) \leq \kappa$. Moreover, $\mathcal{IT}$ ($\mathcal{PT}$) is *fully saturated* , or *maximal*, provided it is κ-saturated for each κ.

Trivially, $\mathcal{IT}$ and $\mathcal{PT}$ are always 0-saturated. Also, κ-saturated implies κ'-saturated for all $\kappa' \leq \kappa$. Moreover, $\mathcal{IT}$ is maximal iff $\mathcal{IT} = Mod\text{-}R \setminus \mathcal{P}$. Similarly, $\mathcal{PT}$ is maximal iff $\mathcal{PT} = Mod\text{-}R \setminus \mathcal{I}$. Since each of these conditions is equivalent to the assertion "$Ext_R(M, N) \neq 0$ whenever M is non-projective and N is non-injective", the two maximality conditions are equivalent.

Most of this section deals with the structure of rings R such that $\mathcal{IT}$ is κ-saturated for $\kappa = 1$, $\kappa = \aleph_0$, and for all κ. Our results show that even the condition of $\mathcal{IT}$ being 1-saturated imposes very strong restrictions on the structure of R. That is, "almost" no R satifies this condition. By Baer's criterion, this means that *for "almost" all rings R there exist right ideals I and J and a module N such that J is not a summand of R, $Ext_R(R/I, N) \neq 0$, but $Ext_R(R/J, N) = 0$.*

Our first result says that if $\mathcal{IT}$ contains all cyclic non-projective modules, then R has no uncountably generated right ideals:

Theorem 4.2. *Let R be a ring such that $\mathcal{IT}$ is 1-saturated. Then each right ideal is countably generated. Moreover, either*
(i) R is right noetherian (i.e. each right ideal is finitely generated), or
(ii) R is von Neumann regular.

Proof. (i) Assume R is not von Neumann regular. Then there is a principal right ideal rR which is not a summand in R. Let $(I_\alpha; \alpha < \kappa)$ be any system of injective

modules. Put $M = \oplus \sum_{\alpha<\kappa} I_\alpha$. We show that M is injective. First, we prove that $Ext_R(R/rR, M) = 0$. Let $\phi \in Hom_R(rR, M)$. Then there is a finite set $F \subseteq \kappa$ such that $Im(\phi) \subseteq \oplus \sum_{\alpha\in F} I_\alpha = M_F$. Since M_F is injective, there is some $\varphi \in Hom_R(R, M_F)$ such that $\varphi \restriction rR = \phi$. Thus, $Ext_R(R/rR, M) = 0$. Since R/rR is an i-test module, M is injective. Hence, a direct sum of any system of injective right modules is injective, and (i) holds.
(ii) Assume R is von Neumann regular, but not right noetherian. Then R is not semisimple and there is an infinite set, $\{e_n; n < \aleph_0\}$, of orthogonal idempotents in R. Put $J = \oplus \sum_{n<\aleph_0} e_n R$ and $M = R/J$. Then J is projective and M is a cyclic non-projective module.
Let I be an injective module and K be a submodule of I. Put $N = I/K$. Since the sequence $0 \to J \to R \to M \to 0$ is exact, we have

$$0 = Ext_R(J, K) \to Ext^2_R(M, K) \to Ext^2_R(R, K) = 0,$$

and $Ext^2_R(M, K) = 0$. Since the sequence $0 \to K \to I \to N \to 0$ is exact, we have

$$0 = Ext_R(M, I) \to Ext_R(M, N) \to Ext^2_R(M, K) = 0,$$

and $Ext_R(M, N) = 0$. As M is i-test, we infer that $N = I/K$ is injective. This proves that any factor module of an injective module is injective, and R is right hereditary.
Now, we shall show that each right ideal is countably generated. On the contrary, assume there is a right ideal I such that $gen(I) > \aleph_0$. Since R is right hereditary, $I = \oplus \sum_{\alpha<\kappa} x_\alpha R$ for an uncountable cardinal κ and some $0 \neq x_\alpha \in R$, $\alpha < \kappa$. Put $J = \oplus \sum_{\alpha<\aleph_0} x_\alpha R$.
Let M be a non-injective module. Let $H = Hom_R(I, M)$. Put $\lambda_0 = card(H)$, i.e. $H = \{h_\beta; \beta < \lambda_0\}$. By induction, define $\lambda_{n+1} = \lambda_n^+$, $n < \aleph_0$. Put $\lambda = sup_{n<\aleph_0} \lambda_n$. Then λ has cofinality ω. By 3.16, $Ext_R(R/J, M^\lambda/M_\lambda) = 0$. Since R/J is a cyclic non-projective module, M^λ/M_λ is injective, and $Ext_R(R/I, M^\lambda/M_\lambda) = 0$. For each $\nu < \lambda$, denote by π_ν the projection of M^λ onto M. Define $f \in Hom_R(I, M^\lambda/M_\lambda)$ by $f(x_\alpha) = m_\alpha + M_\lambda$, $\alpha < \kappa$, where $m_\alpha \in M^\lambda$ is defined by
$\pi_\nu(m_\alpha) = h_\nu(x_\alpha)$ provided $\nu < \lambda_0$;
$\pi_\nu(m_\alpha) = h_\beta(x_\alpha)$ provided $\nu = \lambda_n + \beta$, $\beta < \lambda_0$, $n < \aleph_0$;
$\pi_\nu(m_\alpha) = 0$ otherwise.
Since $Ext_R(R/I, M^\lambda/M_\lambda) = 0$, there is a $g \in Hom_R(R, M^\lambda/M_\lambda)$ such that $g \restriction I = f$. Hence, there is some $y \in M^\lambda$ such that $y.x_\alpha - m_\alpha \in M_\lambda$, for all $\alpha < \kappa$. For $n < \aleph_0$, put $A_n = \{\alpha < \kappa; y.x_\alpha - m_\alpha \in M_n\}$. Then $A_n \subseteq A_{n+1}$ for all $n < \aleph_0$, and $\kappa = \cup_{n<\aleph_0} A_n$. Clearly, there is a $p < \aleph_0$ such that $card(A_p) > \aleph_0$. Then $\pi_\nu(y.x_\alpha - m_\alpha) = 0$ for all $\lambda_p \leq \nu < \lambda$ and all $\alpha \in A_p$.
Put $K = \oplus \sum_{\alpha\in A_p} x_\alpha R$. We shall prove that $Ext_R(R/K, M) = 0$, i.e. that any $h \in Hom_R(K, M)$ extends into some $h' \in Hom_R(R, M)$. First, there is some $\beta < \lambda_0$ such that $h_\beta \restriction K = h$. Put $\nu_0 = \lambda_p + \beta$. Then $h(x_\alpha) = \pi_{\nu_0}(m_\alpha) = \pi_{\nu_0}(y.x_\alpha)$ for all $\alpha \in A_p$. Define $h' \in Hom_R(R, M)$ by $h'(1) = \pi_{\nu_0}(y)$. Then $h' \restriction K = h$, and $Ext_R(R/K, M) = 0$. Since R/K is a cyclic non-projective module, we infer that M is injective, a contradiction. □

We turn to the case when $\mathcal{IT}$ contains all countably generated non-projective modules:

Theorem 4.3. *Let R be a ring such that $\mathcal{IT}$ is $\aleph_0$-saturated. Then either*
(i) R is right artinian, or
(ii) R is von Neumann regular and each right ideal of R is countably generated.

Proof. Assume that R is not von Neumann regular. Proving indirectly, we show that R is right perfect: otherwise, there exist elements $a_i \in R, i < \aleph_0$, such that $(Ra_i \dots a_0; i < \aleph_0)$ is a strictly decreasing chain of principal left ideals of R. Let 1_i, $i < \aleph_0$ be the canonical basis of the free module $F = R^{(\aleph_0)}$ and let $G = \sum_{i<\aleph_0}(1_i - 1_{i+1} \cdot a_i)R \subseteq F$. Put $M = F/G$. By Bass' lemma, M is a countably generated flat module, but M is not projective. Since R is not von Neumann regular, there exists a non-flat left R-module N. Let C be an injective cogenerator for Mod-$\mathbb{Z}$. Since $Tor_R(M,N) = 0$, we have ([CaEi,IV,Proposition 5.1])

$$Ext_R(M, Hom_{\mathbb{Z}}(N,C)) \simeq Hom_{\mathbb{Z}}(Tor_R(M,N),C) = 0.$$

Since N is not flat in R-Mod and C is a cogenerator for Mod-$\mathbb{Z}$, $Hom_{\mathbb{Z}}(N,C)$ is not injective. Hence, M is not i-test, a contradiction. Therefore, R is right perfect and right noetherian, by 4.2. Thus, R is right artinian, and (i) holds.
If R is von Neumann regular, then 4.2 gives (ii). □

Now, we pause to present the "rare" examples of rings possesing many test modules. The first one is an artinian non-singular ring such that $\mathcal{IT}$ (and $\mathcal{PT}$) is maximal.

Example 4.4. *Let K be a skew-field. Denote by $R = UT_2(K)$ the ring of all upper triangular 2×2 matrices over K. Then R is a (left and right) artinian and (left and right) non-singular ring, and $\mathcal{IT}$ and $\mathcal{PT}$ are fully saturated.*

Proof. R is well-known to be artinian and hereditary. Denote by e and f the orthogonal idempotents of R such that $e_{00} = f_{11} = 1$, and all other entries in e and f are zero. Then $J_0 = eR/Soc(eR)$ and $J_1 = fR$ are - upto isomorphism - the only simple modules. Moreover, J_0 is injective, and $eR \simeq I(J_1)$. Let M be any module. There exist cardinals κ and λ such that $Soc(M) \simeq J_0^{(\kappa)} \oplus J_1^{(\lambda)}$. Since $J_0^{(\kappa)}$ is injective, there are submodules N and P in M such that $M = N \oplus P$, $P \simeq J_0^{(\kappa)}$ and $Soc(N) \simeq J_1^{(\lambda)}$. Then $I(N) \simeq (eR)^{(\lambda)}$ is projective, and so is N. Since $\{e, f\}$ is a complete basic set of idempotents of R, there are cardinals μ and ν such that $N \simeq J_1^{(\mu)} \oplus (eR)^{(\nu)}$. Hence, M is isomorphic to a direct sum of direct powers of the modules J_0, J_1 and eR. If M is non-projective and non-injective, then the direct power of J_0, and of J_1, respectively, is non-zero in this decomposition. Since $Ext_R(J_0, J_1) \neq 0$, the assertion holds true. □

Our second example is again a ring such that $\mathcal{IT}$ is maximal:

Example 4.5. *Let R be a commutative local principal ideal ring. Then $\mathcal{IT}$ and $\mathcal{PT}$ are fully saturated.*

Proof. It is well-known that each module is a direct sum of cyclic modules, and the ideals of R form a chain

$$0 = x^m R \subset x^{m-1} R \subset \dots \subset xR = Rad(R) \subset R,$$

where x is a generator of $Rad(R)$ (see [FuS]). Since $Soc(R)$ is simple, R is a QF-ring.

Hence, we have to prove that $Ext_R(R/x^iR, R/x^jR) \neq 0$ for all $0 < i, j < m$. Define $f \in Hom_R(x^iR, R/x^jR)$ by $f(x^i) = 1 + x^jR$ provided $i + j \leq m$, and by $f(x^i) = x^{i+j-m} + x^jR$ otherwise. Then $f \neq g \restriction x^iR$, for all $g \in Hom_R(R, R/x^jR)$. This proves that $Ext_R(R/x^iR, R/x^jR) \neq 0$. □

The ring R from the previous example is an artinian valuation ring. Also noetherian valuation domains possess many i-test modules:

Example 4.6. *Let R be a noetherian valuation domain which is not a field. Then $\mathcal{IT}$ is n-saturated for each $n < \aleph_0$, but it is not $\aleph_0$-saturated.*

Proof. Since R is an almost maximal valuation domain, each finitely generated module is a direct sum of cyclic modules ([FuS]). Hence, it suffices to prove that each cyclic non-projective module is i-test. By the premise, the ideals of R form a chain

$$0 = \cap_{n<\aleph_0} x^nR \subset \dots \subset x^{n+1}R \subset x^nR \subset \dots x^2R \subset xR = Rad(R) \subset R.$$

Let N be a module. Assume there is some $0 < n < \aleph_0$ such that $Ext_R(R/x^nR, N) = 0$. Since $Ann(x^n) = 0$, each element of N is divisible by x^n. Then each element of N is divisible by x^m and $Ext_R(R/x^mR, N) = 0$, for all $0 < m < \aleph_0$. By Baer's criterion, N is injective. Since R is not right artinian, the last assertion follows from 4.3. □

Note that 4.6 shows that 4.2 and 4.3 apply to different classes of rings. Another example of this fact is

Example 4.7. *Let K be a universal differential field of characteristic 0 with differentiation D (i.e. $char(K) = 0$; for each $n < \aleph_0$, each polynomial equation in indeterminates $x_0 = x$, $x_1 = D(x)$, ..., $x_{n-1} = D^{n-1}(x)$ has a solution in K; and each homogenous linear D-differential equation has a non-trivial solution in K). Denote by $R = K[y, D]$ the ring of all differential polynomials in one indeterminate y over K (i.e. the elements of R are polynomials from $K[y]$ with usual addition, and with multiplication given by the identity $ya = ay + D(a)$ and its consequences). Then $\mathcal{IT}$ is n-saturated for each $n < \aleph_0$, but it is not $\aleph_0$-saturated.*

Proof. We shall need several well-known properties of R (proved in [Fa1], [CzFa] and [K]): first, R is a simple non-commutative principal right ideal domain. Moreover, all simple modules are isomorphic to a simple module J, and J is injective. Let K be a right ideal of R. Since R has a right division algorithm, R/K is a semisimple module. Let Q be the right skew field of quotients of R. Then Q is an injective module. Since R is right noetherian, each injective module is isomorphic to a direct sum of copies of Q and J. In particular $I(N)/N$ is isomorphic to a direct power of J for each module N. If F is a finitely generated module with $Soc(F) = 0$, then F is flat, whence F is free.
Now, we prove that each finitely generated non-projective module F is i-test: Assume $Ext_R(F, M) = 0$ for a module M. Note that $F = Soc(F) \oplus G$, where G is finitely generated and $Soc(G) = 0$. Hence G is free, and $Soc(F) \neq 0$. Similarly, $M = Soc(M) \oplus N$, where $Soc(N) = 0$ and $I(N) \simeq Q^{(\kappa)}$ for a cardinal κ. W.l.o.g., we can assume that $N \neq 0$. We have $Ext_R(J, N) = 0$ and $Hom_R(J, Q^{(\kappa)}) = 0$. Then also $Hom_R(J, I(N)/N) = 0$. Since $I(N)/N$ is isomorphic to a direct power of J, we infer that $N = I(N)$.

The last assertion is a consequence of 4.3. $\square$

In fact, the proof of the last assertions of 4.6 and 4.7 is constructive. Taking the module $M = F/G$ as in the proof of 4.3, we obtain a particular countably generated non-projective module which is not i-test.

Now, we proceed with the structure theory and show that the “rare” examples of 4.4-4.7 are in a sense typical. First, we have

Lemma 4.8. *Let R be a ring such that $\mathcal{IT}$ is 1-saturated. Then all non-projective simple modules are isomorphic.*

Proof. Let J be a non-projective simple module and N be a non-injective module. By the premise, $Ext_R(J, N) \neq 0$, and $Hom_R(J, I(N)/N) \neq 0$. Hence, the module $I(N)/N$ has a (transfinite) composition series with factors isomorphic to J. Since N was arbitrary, all non-projective simple modules are isomorphic to J. $\square$

The following theorem shows that we can restrict our investigation to indecomposable rings:

Theorem 4.9. *Let κ be a cardinal. Let R be a ring such that $\mathcal{IT}$ is κ-saturated (fully saturated). Then either*
(i) R is an indecomposable ring; or
(ii) $R = R' \boxplus R''$, where R'' is a semisimple ring and R' is an indecomposable ring such that the class of all i-test right R'-modules is κ-saturated (fully saturated).

On the other hand, if $R = R' \boxplus R''$ and R', R'' are as in (ii), then $\mathcal{IT}$ is κ-saturated (fully saturated).

Proof. First, note that for any decomposition $R' \boxplus R''$ of the ring R, either R' or R'' is semisimple. Indeed, taking any non-projective simple right R'-module M and any non-injective right R''-module N, we have $Ext_R(M, N) = 0$, a contradiction.
Let B be a representative set of all projective simple modules (possibly, $B = \emptyset$). Let J be a simple non-projective module. By 4.8, $A = B \cup \{J\}$ is a representative set of all simple modules. Let C and D be two disjoint subsets of A. Denote by I_C and I_D the trace of C and of D, respectively, in R. Then I_C and I_D are two-sided ideals of R. Moreover, $Hom_R(I_C, I_D) = 0$, and $Ext_R(R/I_C, I_D) = 0$. Hence, either I_C is a summand of R, or I_D is injective (and a summand of R).
Assume C and D are two infinite disjoint subsets of A. Then neither I_C nor I_D is finitely generated, a contradiction. Hence, A is finite. This implies that either R is indecomposable, or R has a decomposition $R = R' \boxplus R''$, where R'' is semisimple and R' is indecomposable. The final assertion follows from the fact that

$$Ext_R(M, N) \simeq Ext_{R'}(MR', NR') \oplus Ext_{R''}(MR'', NR'')$$

whenever $M, N \in Mod\text{-}R$. $\square$

Proposition 4.10. *Let R be an indecomposable ring such that $\mathcal{IT}$ is 1-saturated. Clearly, either (I) all simple modules are isomorphic, or (II) there are at least two non-isomorphic simple modules.*
In the case (I), either

(Ia) R is isomorphic to a full matrix ring over a local right artinian ring, or

(Ib) R is a simple ring such that each right ideal is countably generated,

or

(Ic) R is right noetherian, right non-singular and non-right perfect, and $0 = Soc(R) \subset Rad(R)$.

In the case (II), R is right semiartinian and right hereditary and, upto isomorphism, there exist two simple modules J and P. Moreover, J is Σ*-injective and non-projective and P is projective.*

Proof. Take a simple non-projective module J. If B is a representative set of all non-projective modules, then $A = B \cup \{J\}$ is a representative set of all simple modules by 4.8. For $C \subseteq A$, denote by I_C the trace of C in R. We distinguish the following cases:

(Ia) $B = \emptyset$ and $I_J \neq 0$. Then R is not von Neumann regular, and R is right noetherian by 4.2. Since J embeds into R, we have $Soc(R) = I_J \neq 0$. Moreover, if $Soc(R/Soc(R)) = 0$, then $Hom_R(I_J, N) = 0$ for any submodule N of $R/Soc(R)$. Hence, $Ext_R(R/I_J, N) = 0$, N is injective, and $R/Soc(R)$ is completely reducible, a contradiction. Similarly, it follows that R has a (finite) socle sequence, and R is right artinian. This implies that $R \simeq (eR)^{(n)}$ for some $n < \aleph_0$ and some indecomposable idempotent $e \in R$. Then $R \simeq Hom_R((eR)^{(n)}, (eR)^{(n)}) \simeq M_n(eRe)$, where eRe is a local right artinian ring.

(Ib) $B = \emptyset$, $I_J = 0$ and $Rad(R) = 0$. Let $I \neq R$ be a two-sided ideal of R. Let M be a maximal right ideal containing I. Then $J \simeq R/M$ and $I.J = 0$. Since $Rad(R) = 0$, we have $Ann(J) = 0$ and $I = 0$. Hence, R is a simple ring.

(Ic) $B = \emptyset$, $I_J = 0$ and $Rad(R) \neq 0$. By 4.2, R is right noetherian. Also $Soc(R) = I_J = 0$. Assume R is right perfect. Then R is right artinian and $Soc(R) = 0$, a contradiction. Further, assume $Sing(R) \neq 0$. Take $0 \neq r \in R$ such that $K = Ann(r) \trianglelefteq R$. Then $R/K \simeq rR$ is a submodule of $I(K)/K$. Since $Ext_R(J, K) \neq 0$, also $Hom_R(J, I(K)/K) \neq 0$, and $I(K)/K$ has a transfinite composition series with factors isomorphic to J. In particular, J embeds into $R/K \simeq rR \subset R$, a contradiction. Hence, R is right non-singular.

(II) $B \neq \emptyset$. Then $I_B \neq 0$. Since $Hom_R(I_B, I_J) = 0$ and R is indecomposable, we infer that $Ext_R(R/I_B, I_J) = 0$ and $I_J = 0$. Take $P \in B$ and put $C = B \setminus \{P\}$. Similarly, we get $I_C = 0$, i.e. $C = \emptyset$ and $A = \{P, J\}$. Take a cardinal κ and consider the module $N = J^{(\kappa)}$. Since $Hom_R(I_P, N) = 0$, we have $Ext_R(R/I_P, N) = 0$, and N is injective.

Let M be an arbitrary module. Denote by M_P the trace of P in M. Let N be a submodule of M/M_P. Since $Hom_R(I_P, N) = 0$ we have $Ext_R(R/I_P, N) = 0$, N is injective and M/M_P is semisimple. Hence, M/M_P is isomorphic to a direct power of J. In particular, R is right semiartinian.

Proving indirectly, assume R is not right hereditary. Then there are an injective module I and a submodule K of I such that $N = I/K$ is not injective. Put $M = R/I_P$. Since the sequence $0 \to I_P \to R \to M \to 0$ is exact and I_P is projective, we get $0 = Ext_R(I_P, K) \to Ext^2_R(M, K) \to Ext^2_R(R, K) = 0$, and $Ext^2_R(M, K) = 0$. Since the sequence $0 \to K \to I \to N \to 0$ is exact, we have $0 = Ext_R(M, I) \to Ext_R(M, N) \to Ext^2_R(M, K) = 0$, and $Ext_R(M, N) = 0$. Then M is not i-test, a contradiction. □

Note that all the possibilities from 4.10 do occur: (Ia) in Example 4.5, (Ib) in 4.7, (Ic) in 4.6, and (II) in 4.4.

Using more involved arguments, we shall obtain a better characterization of rings of type (II). As a first step, we have

Lemma 4.11. *Any ring of type (II) is right artinian.*

Proof. Since R is right semiartinian and 4.2 holds, it suffices to prove that R is not von Neumann regular. On the contrary, assume R is von Neumann regular. By 4.10(II), $R/I_P \simeq J^{(n)}$ for some $n < \aleph_0$. Hence, $\bar{R} = R/I_P$ is a simple artinian ring and there is a complete set of orthogonal idempotents $\{\bar{e}_0, \dots, \bar{e}_{n-1}\} \subseteq \bar{R}$ such that $\bar{e}_i\bar{R}$ is a minimal right ideal of $\bar{R}$ for all $i < n$. Since R is von Neumann regular, this set can be lifted modulo I_P into a complete set of orthogonal idempotents, $\{e_0, \dots, e_{n-1}\}$, of the ring R so that $e_i + I_P = \bar{e}_i$ for all $i < n$. Since R is not semisimple, $dim(Soc(R)) = dim(I_P) = \kappa$ for some $\kappa \geq \aleph_0$. Hence, $Soc(R) = \oplus \sum_{i<n} Soc(e_i R)$ and there is some $e = e_i$ such that $dim(Soc(eR)) = \kappa$. Since R is von Neumann regular, there are a complete decomposition $\oplus \sum_{\alpha<\kappa} s_\alpha R$ of $Soc(R)$ and a subset $A \subseteq \kappa$ such that

(1) $0 \in A$, $card(A) = \kappa$,
(2) $f = s_0$ is an idempotent of R such that $efe = f$,
(3) $Soc((e-f)R) = \oplus \sum_{\alpha \in A, \alpha \neq 0} s_\alpha R$, and
(4) $Soc((1-e)R) = \oplus \sum_{\alpha \notin A} s_\alpha R$.

By 4.10(II), $Soc(R) \trianglelefteq R$ and $s_\alpha R \simeq P$ for all $\alpha < \kappa$. Hence, there is a canonical ring isomorphism $\phi : End_R(Soc(R)) \simeq CFM_\kappa(K)$, where $K = End_R(P)$. Since R is right non-singular, the canonical mapping $\varphi : End_R(I(R)) \to End_R(Soc(R))$ defined by $\varphi(x) = x \restriction Soc(R)$ is a ring isomorphism. Denote by Q the maximal right quotient ring of R. Then $Q = I(R)$ (as modules), and the canonical mapping $\psi : Q \to End_R(I(R))$ given by $h(q)(1) = (1)q$ is a ring isomorphism.

W.l.o.g., we can view R as a submodule of Q. Moreover, since $\pi = \phi\varphi\psi$ is a ring isomorphism, we can identify Q with $CFM_\kappa(K)$, whence R becomes a subring of $CFM_\kappa(K)$. By (2), (3) and (4), $m = \pi(e)$ is a matrix such that $m_{\alpha\alpha} = 1$ provided $\alpha \in A$, and $m_{\alpha\beta} = 0$ otherwise. If $s \in Soc(R)$, then $\psi(qs)(1) = (1)(qs) = ((1)q)s = (s)q = s' \in Soc(R)$, whence $\psi(qs) = \psi(s')$ for all $q \in Q$. Thus, $Soc(R)$ is a left ideal of the ring $CFM_\kappa(K)$. Moreover, if $s \in Soc(R)$, $s = \sum_{\alpha \in F} s_\alpha r_\alpha$ for a finite set $F \subset \kappa$, then $\pi(s)$ is a matrix which is zero in any row indexed by $\alpha \in \kappa \setminus F$. Since $\pi(s)$ is also column finite, it has only finitely many non-zero entries. Further, by (2), $(\pi(f))_{00} = 1$ and $(\pi(f))_{\alpha\beta} = 0$ otherwise. Define $\{f_n, n < \aleph_0\} \subset Q$ by $f_0 = f$, and $(f_n)_{n0} = 1$, $(f_n)_{\alpha\beta} = 0$ otherwise, for $0 < n < \aleph_0$. Since $Soc(R)$ is a left ideal of Q, we infer that $f_n \in Soc(R)$ for all $n < \aleph_0$.

We shall construct a cyclic non-projective module M which is not i-test: Consider the right ideal $I = \oplus \sum_{n<\aleph_0, n \text{ even}} (f_n - f_{n+1})R$. Since I is not finitely generated, $M = R/I$ is not projective.

It remains to construct a non-injective module N such that $Ext_R(M, N) = 0$. Let $\lambda = card((eRe + Soc(R))/Soc(R))$. Then $(eRe + Soc(R))/Soc(R) = \{r_\beta + Soc(R), \beta < \lambda\}$ for some $r_\beta \in eRe$, $\beta < \lambda$, and $r_0 = 0$. Put $L = Q^{(\lambda)}$ and denote by π_β, $\beta < \lambda$, the β-th projection of L onto Q. Note that $Soc(L) = (Soc(R))^{(\lambda)} \trianglelefteq L \trianglelefteq I(L)$, and $I(L)/Soc(L)$ is a completely reducible module. W.l.o.g., we shall view Q as a submodule of L consisting of all $l \in L$ such that $\pi_\beta(l) = 0$ for all $0 < \beta < \lambda$. Consider the matrix $q \in Q$ defined by $q_{i,i+1} = q_{i+1,i} = 1$ provided $i < \aleph_0$, i even, and by $q_{\alpha\beta} = 0$ otherwise. Note that $q = eqe$ and $e + q \notin Soc(R)$, as $e + q$ has infinitely many non-zero entries. Moreover, take $r \in R$ such that $(e+q)r \in Soc(R)$. Then $(e+q)ere \in Soc(R)$. Assume $(eRe)r \nsubseteq Soc(R)$. Then $(er'e)(ere) = e + s$, for some $r' \in R$ and $s \in Soc(R)$. Since $\bar{e}\bar{R}\bar{e} \simeq End_{\bar{R}}(\bar{e}\bar{R}) \simeq End_R(J)$ is a skew-field,

also $(ere)(er'e) = e + s'$ for some $s' \in Soc(R)$, whence $e + q = (e+q)(erer'e - s') \in Soc(R)$, a contradiction. This implies that there exists a maximal submodule N of $I(L)$ such that $Soc(L) \subseteq N$, $e \notin N$, and $q_\beta \in N$ for all $0 < \beta < \lambda$. Here, q_β denotes the element of L defined by $\pi_0(q_\beta) = r_\beta$, $\pi_\beta(r_\beta) = e + q$, and $\pi_{\beta'}(r_\beta) = 0$ otherwise. Since $N \neq I(L)$ and $Soc(L) \trianglelefteq N \trianglelefteq I(L)$, N is not injective.
Finally, let $\phi \in Hom_R(I, N)$. Then there is some $x \in I(L)$ with $xe(f_n - f_{n+1}) = \phi(f_n - f_{n+1})$ for all $n < \aleph_0$. Since N is a maximal submodule of $I(L)$, we have $eR + N = I(L)$, and $x = er + y$ for some $r \in R$ and $y \in N$. Then $ere = r_\beta$ for some $\beta < \lambda$. If $r_\beta \neq 0$ (i.e. if $\beta > 0$), then $r_\beta(f_n - f_{n+1}) = q_\beta(f_n - f_{n+1})$, whence $(q_\beta - ye)(f_n - f_{n+1}) = xe(f_n - f_{n+1}) = \phi(f_n - f_{n+1})$. Define $\varphi \in Hom_R(R, N)$ by $\varphi(1) = q_\beta + ye$ provided $\beta > 0$, and by $\varphi(1) = ye$ otherwise. Then $\varphi \restriction I = \phi$, and $Ext_R(R/I, N) = 0$. □

Now, we introduce a class of rings which plays a crucial role in characterizing rings of type (II):

Definition 4.12. Let $0 < m < \aleph_0$. Let S and T be skew-fields such that T is a subring of $M_m(S)$. Denote by $^-$ the mapping from $M_{m+1}(S)$ to $M_m(S)$ defined by $(a^-)_{ij} = a_{ij}$ for all $a \in M_{m+1}(S)$ and $i, j < m$. Define $R = UT(m, S, T)$ as the subring of $M_{m+1}(S)$ consisting of all matrices $a \in M_{m+1}(S)$ satisfying
(1) $a_{mi} = 0$ for all $i < m$, and
(2) $a^- \in T$.

Note that the rings $UT(m, S, T)$ include the following important particular cases:
(1) upper triangular matrix rings of degree two over skew-fields (as $UT(1, K, K) = UT_2(K)$ for any skew-field K);
(2) the rings $UT(1, S, T)$ where T is a skew field which is a proper subring of the skew-field S (this example, in the particular case when S is a quadratic extension of T, will be essential in §5);
(3) the rings $UT(m, S, T)$ for which $m > 1$, S is commutative, and the skew-field T contains a copy of S (for example, if $\mathbb{C}$ is the field of all complex numbers, $\mathbb{H}$ the skew-field of all quaternions and φ the canonical ring embedding of $\mathbb{H}$ into $M_2(\mathbb{C})$, then $UT(2, \mathbb{C}, \varphi(\mathbb{H}))$ is a subring of $M_3(\mathbb{C})$).

Basic properties of the rings $UT(m, S, T)$ can easily be described:

Lemma 4.13. *Let $R = UT(m, S, T)$. For each $i \leq m$, denote by e_i the matrix from R defined by $(e_i)_{im} = 1$ and $(e_i)_{jk} = 0$ otherwise. Put $e = e_m$, $f = 1 - e$, $P = eR$ and $J = R/Soc(R)$.*
(i) If I is a proper right ideal of R, then either $I = fR$ or $I \subseteq \oplus \sum_{i \leq m} e_i R$.
(ii) $Soc(R) = \oplus \sum_{i \leq m} e_i R$ and $Rad(R) = \oplus \sum_{i < m} e_i R$.
(iii) The mapping $\psi : R/Soc(R) \to T$ defined by $\psi(r + Soc(R)) = r^-$ is a ring isomorphism.
(iv) R is an indecomposable right hereditary right artinian basic ring. The set $\{e, f\}$ is a complete basic set of idempotents of R.
(v) $\{P, J\}$ is a representative set of all simple modules. Moreover, P is projective, but not injective, while J is $\sum$-injective, but not projective.
(vi) $I(R) = M_{m+1}(S)$, and the maximal right quotient ring of R is isomorphic to $M_{m+1}(S)$.

Proof. By easy matrix computations. □

As a further step of the characterization, we have

Theorem 4.14. *Any ring of type (II) is Morita equivalent to some $UT(m, S, T)$.*

Proof. Let R be a ring of type (II). By 4.10(II), we have $0 \subset Rad(R) \subset Soc(R) = I_P$, where P is (upto isomorphism) the only projective module. Moreover, by 4.11, there is a complete orthogonal set, $\{e_0, \dots, e_k, e'_0, \dots, e'_l\}$, of primitive idempotents of R such that $P \cong e_i R$ for all $i \le k$ and $e'_j R \cong e'_{j'} R$ for all $j, j' \le l$. By 4.10(II), $Soc(R) \cong P^{(n)}$ for some $k < n < \aleph_0$, and $R/Soc(R)$ is a simple artinian ring. Put $S = End_R(P)$. Then S is a skew-field. Let R' be the basic ring of R. Since R is Morita equivalent to R', it suffices to show that R' is ismorphic to some $UT(m, S, T)$.
Clearly, $R' = (e+f)R(e+f)$, where $e = e_0$ and $f = e'_0$, and $\{e, f\}$ is a basic set of primitive idempotents of R'. Let $m = dim(Soc(fR'))$. The same argument as in the proof of 4.10(II) shows that R' is (canonically isomorphic to) a subring of the full matrix ring $Q = M_{m+1}(S)$ so that $e_{mm} = 1$, and $e_{jj'} = 0$ otherwise. Moreover, as in the proof of 4.10(II), we see that $Soc(R')$ is a left ideal of Q. In particular, each of the matrices x_i, $i \le m$, defined by $(x_i)_{im} = 1$ and by $(x_i)_{jj'} = 0$ otherwise, belongs to R'. Put $X = \{q \in Q; q_{ij} = 0 \text{ for all } i \le m \text{ and } j < m\}$. Then $X \subseteq Soc(R')$. Since f is a primitive idempotent, we have $Rad(R') = Rad(fR') = Soc(fR')$. Hence, $x_i \in Rad(R')$ for each $i < m$. Since $Rad(R').Soc(R') \subseteq Rad(Soc(R')) = 0$, we have $X = Soc(R')$. In particular, for each $r \in R'$ and each $i < m$, we have $x_i r \in Soc(R')$, whence $r_{mi} = 0$ for all $i < m$. If $q \in Q$, define $q^- \in M_m(S)$ as in 4.12. Let $T = \{q^-; q \in R\}$. Then T is a subring of $M_m(S)$ such that $T \cong R'/Soc(R') \cong fR'/Rad(fR')$ is a skew-field. By 4.12, $R' \cong UT(m, S, T)$. □

The rings Morita equivalent to $UT(m, S, T)$ are completely characterized as certain block upper triangular matrix rings:

Theorem 4.15. *Let $\bar{R} = UT(m, S, T)$. A ring R is Morita equivalent to $\bar{R}$ if and only if there are $0 < n < \aleph_0$ and $0 < p < \aleph_0$ such that R is isomorphic to a subring of $M_{m.n+p}(S)$ consisting of all matrices of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where $A \in M_n(T) \subseteq M_{m.n}(S)$, $B \in M_{m.n \times p}(S)$ and $C \in M_p(S)$.*

Proof. By a direct matrix computation, using the fact that R is Morita equivalent to $\bar{R}$ iff there are $0 < q < \aleph_0$ and an idempotent matrix $e \in M_q(\bar{R})$ such that $M_q(\bar{R})eM_q(\bar{R}) = M_q(\bar{R})$ and $R \cong eM_q(\bar{R})e$. □

In general, we do not know which of the matrix rings from 4.15 are of type (II). Nevertheless, we have a complete answer for the most important cases of 4.12(1)-(3).
First, recall that the property of $\mathcal{IT}$ being κ-saturated is preserved by adding a semisimple ring to R (see 4.9). The following lemma shows that this property is also preserved by Morita equivalence:

Lemma 4.16. *Let λ be an infinite cardinal. Let R be a ring such that $\mathcal{IT}$ is κ-saturated for all $\kappa < \lambda$ (fully saturated). Let $\bar{R}$ be Morita equivalent to R. Then the class of all i-test right $\bar{R}$-modules is κ-saturated for all $\kappa < \lambda$ (fully saturated).*

Proof. This follows from 4.1 and from the fact that the property "to be κ-generated" is Morita invariant for each $\kappa \ge \aleph_0$. □

Putting together 4.4 and 4.16, we obtain an answer for the case 4.12(1):

Example 4.17. *Let S be a skew-field. Let R be a ring Morita equivalent to $UT_2(S)$. Then R is isomorphic to the ring from 4.16, with $m = 1$ and $T = S$. Moreover, the classes $\mathcal{IT}$ and $\mathcal{PT}$ are fully saturated. In particular, R is of type (II).*

Now, we turn to the case 4.12(2):

Example 4.18. *Let S and T be skew-fields such that T is a subring of S. Then the ring $R = UT(1, S, T)$ is of type (II).*

Proof. Let M be a cyclic non-projective module and let N be a module such that $Ext_R(M, N) = 0$. We shall prove that N is injective.
In view of 4.13(iv), w.l.o.g. we assume that M is indecomposable and that M has a projective cover. By 4.13(i) and (ii), this implies that $M \cong fR/Rad(R) \cong J$. Put $r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$. Then $rR = Rad(R)$. By 4.13(v), w.l.o.g. we assume that the trace of J in N is 0. Hence, there is a cardinal $\kappa > 0$ such that

$$P^{(\kappa)} \cong \{\begin{pmatrix} 0 & d \end{pmatrix}; d \in CFM_{\kappa\times 1}(S)\} = Soc(N) \subseteq N \subseteq I(N) = \\ = \{\begin{pmatrix} d' & d \end{pmatrix}; d, d' \in CFM_{\kappa\times 1}(S)\}.$$

Take an arbitrary $a \in CFM_{\kappa\times 1}(S)$. Define $\varphi \in Hom_R(Rad(R), N)$ by $\varphi(r) = \begin{pmatrix} 0 & a \end{pmatrix}$. Since $Ext_R(M, N) = 0$, φ extends to some $\phi \in Hom_R(fR, N)$, i.e. there is some $x \in N$ with $xf = f$ and $xr = \varphi(r)$. This implies that $x = \begin{pmatrix} a & 0 \end{pmatrix}$. Since a was arbitrary, we infer that that $N = I(N)$, i.e. N is injective. □

On the other hand, the rings from 4.12(3) are never of type (II):

Proposition 4.19. *Let $R = UT(m, S, T)$ and assume that $m > 1$, S is commutative and T contains a copy of S. Then there is a non-projective cyclic module M which is not i-test.*

Proof. We shall use the notation of 4.13. Put $M = fR/rR$, where $r \in Rad(R)$ is defined by $r_{0m} = 1$, and $r_{ij} = 0$ otherwise. Since $r \in Rad(fR)$, M is not projective.
Let X be a maximal right T-subspace of the right T-space $M_m(S)$. Denote by N_X the submodule of $Q = M_{m+1}(S)$ consisting of all matrices $n \in Q$ such that $n^- \in X$ and $n_{mi} = 0$ for all $i \leq m$. Then N_X is a maximal submodule of the injective module Q and $Soc(N_X) = Soc(Q)$. In particular, N_X is not injective.
Put $A = M_{m\times 1}(S)$ and take $r' \in A$ such that $r'_{00} = 1$, and $r'_{i0} = 0$ otherwise. First, assume $T.r' = A$. Take any maximal right T-subspace X of $M_m(S)$ such that $T \subseteq X$. Then any homomorphism from rR to N_X extends to a homomorphism from fR to N_X, whence $Ext_R(M, N_X) = 0$. Now, assume $B = T.r' \subset A$, i.e. B is a proper left T-subspace of A. Since T contains a copy of S and S is commutative, B is also a proper (left, right) S-subspace of A. Hence, there is some $0 \neq y \in M_m(S)$ with $y.B = 0$. Take a maximal right T-submodule X of $M_m(S)$ such that $X \oplus yT = M_m(S)$ in Mod-T. Then $A = M_m(S).r' = (X \oplus yT).r' = X.r' + y.B = X.r'$. This implies that any homomorphism from rR to N_X extends to a homomorphism from fR to N_X, and $Ext_R(M, N_X) = 0$. □

We sum up our results for the case when $\mathcal{IT}$ is 1-saturated. To simplify notation, we shall write $R = R'(\boxplus R'')$ to denote that either $R = R'$ or $R = R' \boxplus R''$:

Theorem 4.20. *Let R be a ring such that $\mathcal{IT}$ is 1-saturated. Then $R = R'(\boxplus R'')$, where R'' is a semisimple ring and R' is an indecomposable ring such that each cyclic non-projective right R'-module is i-test. Moreover, either*
(I) all simple right R'-modules are isomorphic and either
(Ia) R' is isomorphic to a full matrix ring over a local right artinian ring, or
(Ib) R' is a simple ring such that each right ideal is countably generated, or
(Ic) R' is right noetherian, right non-singular and non-right perfect, and $0 = Soc(R') \subset Rad(R')$,
or
(II) R' is Morita equivalent to some $UT(m, S, T)$.

Proof. By 4.9, 4.10 and 4.14. □

Before proceeding with the structure of rings such that $\mathcal{IT}$ is $\aleph_0$-saturated, we consider the case of $R = UT(m, S, T)$:

Lemma 4.21. *Let $R = UT(m, S, T)$. Then $\mathcal{IT}$ is 2-saturated if and only if $m = 1$ and $T = S$ (i.e. if and only if $R = UT_2(S)$).*

Proof. The "if" part was proved in 4.4. For the "only if" part, assume that $T \neq M_m(S)$. We shall construct a non-projective 2-generated module M and a non-injective module N such that $Ext_R(M, N) = 0$.
We use the notation of 4.13. Since $T \neq M_m(S)$, there is a basis, $\{b_\alpha; \alpha < \kappa\}$, of the right T-module $M_m(S)$ such that $b_0 = 1$ and $\kappa > 1$. Put $M = (fR)^{(2)}/gR$, where $g = (g_0, g_1) \in Soc((fR)^{(2)})$ is such that $(g_0)_{im} = (b_0)_{i,m-1}$ and $(g_1)_{im} = (b_1)_{i,m-1}$ for all $i < m$. Since $Soc(fR) = Rad(fR) << fR$, also $gR << (fR)^{(2)}$. Hence, M is a non-projective 2-generated module.
For each $\alpha < \kappa$, take $c_\alpha \in I(R)/Soc(R)$ such that $c_\alpha = d_\alpha + Soc(R)$ for some $d_\alpha \in I(R)$ and $(d_\alpha)^- = b_\alpha$, $d_0 = 1$. Put $\bar{T} = R/Soc(R)$ and let $\sigma = card(\bar{T})$, i.e. $\bar{T} = \{t_\beta; \beta < \sigma\}$. By 4.13(iii), $\bar{T} \cong T$ is a skew-field. Clearly, $\{c_\alpha; \alpha < \kappa\}$ is a right $\bar{T}$-independent subset of $I(R)/Soc(R)$. By 4.13(vi), there are some $\lambda \geq \kappa$ and $c_\alpha \in I(R)/Soc(R)$, $\kappa \leq \alpha < \lambda$, such that $\{c_\alpha; \alpha < \lambda\}$ is a right $\bar{T}$-basis of $I(R)/Soc(R)$.
Put $I = I(R)^{(1+\sigma)} = CFM_{(m+1)(1+\sigma)\times(m+1)}(S)$. We identify $I(R)$ with the first summand of I. By 4.13(ii), $Soc(I) = Soc(R^{(1+\sigma)})$ consists exactly of those elements of I whose i-th column is zero for each $i < m$. Moreover, $Soc(I) \trianglelefteq I$, $I/Soc(I)$ is isomorphic to a direct power of J, and $I/Soc(I)$ is a right $\bar{T}$-module. For each $\gamma < 1+\sigma$, denote by ν_γ the γ-th canonical embedding of $I(R)/Soc(R)$ into $I/Soc(I)$. Put $A = \{\nu_\gamma c_\alpha; 1 \leq \gamma < 1 + \sigma, \alpha < \lambda, \alpha \neq 1\} \cup \{\nu_0 c_0 t_\beta + \nu_{1+\beta} c_1; \beta < \sigma\}$. Then A is a right $\bar{T}$-independent subset of $I/Soc(I)$. In particular, there is a maximal submodule N of I such that $Soc(N) = Soc(I) \subseteq N \subset I$, $\nu_0 c_0 \notin N/Soc(I)$ and $A \subseteq N/Soc(I)$. Since $N \trianglelefteq I$ and $N \neq I$, N is not injective. Note that $\rho(N) \oplus \rho(R) = I/Soc(I)$ in Mod-$\bar{T}$, $\rho : I \to I/Soc(I)$ being the projection.
We shall prove that $Ext_R(M, N) = 0$. Denote by $\pi : I \to I/N$ the projection. Then $Ext_R(M, N) \cong Hom_R(M, I/N)/Im(Hom_R(M, \pi))$. Take an arbitrary $\phi \in Hom_R(M, I/N)$. We have to find a $\varphi \in Hom_R(M, I)$ such that $\phi = \pi\varphi$. Of course, $\phi((f, 0) + gR) = x_0 + N$ and $\phi((0, f) + gR) = x_1 + N$ for some $x_0, x_1 \in I$ such that $x_0 g_0 + x_1 g_1 \in Soc(N)$. By the choice of A and N, we have $\rho(N) + \rho(N.d_1) \supseteq \rho(R)$, whence $N + N.d_1 = I$. In particular, there exist $n_0, n_1 \in N$ such that $n_0 g_0 + n_1 g_1 = x_0 g_0 + x_1 g_1$. Define $\varphi \in Hom_R(M, I)$ by $\varphi((f, 0) + gR) = x_0 - n_0$ and $\varphi((0, f) + gR) = x_1 - n_1$. Then $\pi\varphi = \phi$. This proves that $Ext_R(M, N) = 0$. □

Theorem 4.22. *Let R be a ring such that $\mathcal{IT}$ is $\aleph_0$-saturated. Then $R = R'(\boxplus R'')$, where R'' is a semisimple ring and R' is an indecomposable ring such that each countably generated non-projective right R'-module is i-test. Moreover, either*

(Ia) R' is isomorphic to a full matrix ring over a local right artinian ring, or

(Ib) R' is a simple von Neumann regular ring such that all right ideals are countably generated and all simple right R'-modules are isomorphic, or

(II) R' is Morita equivalent to $UT_2(S)$ for a skew-field S.

Proof. By 4.3, 4.16, 4.20 and 4.21. □

Corollary 4.23. *Let R be a right non-singular ring such that R is not von Neumann regular. Then the following conditions are equivalent:*
(i) $\mathcal{IT}$ is $\aleph_0$-saturated,
(ii) $\mathcal{IT}$ is fully saturated,
(iii) $\mathcal{PT}$ is fully saturated,
(iv) $R = R'(\boxplus R'')$, where R'' is a semisimple ring and there is a skew-field S such that R' is Morita equivalent to $UT_2(S)$.

Though 4.22-3 provide a complete characterization in the type (II) case, the type (I) case is still open. For example, so far, no examples of von Neumann regular rings satisfying the condition (Ib) are known. There are surely no examples of size less than continuum:

Proposition 4.24. *Let R be a simple von Neumann regular ring with $card(R) < 2^{\aleph_0}$. Then there are uncountably many non-isomorphic simple modules.*

Proof. First, we define a 2-branching tree $(T, <)$ of height ω as follows: $T = \cup_{n<\aleph_0} T_n$, where T_n is the n-th level of T and T_n consists of a complete set of orthogonal idempotents of R defined by induction as follows: $T_0 = \{1\}$; if $e \in T_n$, then $ReR = R$, i.e. the rings eRe and R are Morita equivalent, whence there are orthogonal idempotents $f_e, g_e \in R$ such that $f_e \neq e \neq g_e$ and $e = f_e + g_e$, and we put $T_{n+1} = \cup_{e\in T_n}\{f_e, g_e\}$. Since T_n is a complete set of idempotents, so is T_{n+1}. If $n < \aleph_0$, $e \in T_n$ and $e' \in T_{n+1}$, we define $e \prec e'$ iff either $e' = f_e$ or $e' = g_e$. Now, $<$ is defined as the transitive closure of $\prec$ on T. Denote by B the set of all branches of T. Clearly, $card(B) = 2^{\aleph_0}$. For each $b \in B$, define a right ideal I_b of R by $I_b = \sum_{e\in(T\setminus b)} eR$. Then $I_b \neq R$, and there is a maximal right ideal J_b of R such that $I_b \subseteq J_b$.
Let M be a simple module. Put $B_M = \{b \in B; R/J_b \cong M\}$. Since R is a simple ring, the Jacobson density theorem implies that we can view R as a subring of $End_K(M)$, where $K = End_R(M)$ is a skew-field. Since M is isomorphic to a factor module of R, also $card(M) < 2^{\aleph_0}$ and $dim_K(M) < 2^{\aleph_0}$. Let $b \in B_M$. Then $Hom_R(R/J_b, M) \neq 0$ and there is some $0 \neq m_b \in M$ such that $m_b I_b = 0$.
We shall prove that the set $U_M = \{m_b; b \in B_M\}$ is a K-independent subset of M. On the contrary, let $\{m_{b_i}; i < n\}$ be a K-dependent subset of U_b having a minimal cardinality, $n > 1$. Then $\sum_{i<n} k_i m_{b_i} = 0$ for some $0 \neq k_i$, $i < n$. Since all the branches b_i, $i < n$, are different, there is some $e \in b_0 \setminus \cup_{0<i<n} b_i$. Take $p < \aleph_0$ such that $e \in T_p$. Since T_p is complete, $1 - e$ is a sum of some elements of $T \setminus b_0$. Then $0 = \sum_{i<n} k_i m_{b_i}(1-e) = \sum_{0<i<n} k_i m_{b_i}$, in contradiction with the minimality of n.
Finally, denote by $\mathcal{S}$ a representative set of all simple modules. Clearly, $B = \cup_{M\in\mathcal{S}} B_M$. Since $card(B_M) \leq dim_K(M) < 2^{\aleph_0}$ for each $M \in \mathcal{S}$, we infer that $card(\mathcal{S}) \geq cf(2^{\aleph_0}) > \aleph_0$. □

Corollary 4.25. *Let R be a right non-singular ring such that $card(R) < 2^{\aleph_0}$. Then the conditions (i) - (iv) of 4.23 are equivalent.*

By 4.5, artinian valuation rings provide examples of rings of type (Ia). In fact, the description of type (Ia) gets closer to the valuation ring case if we assume that $\mathcal{IT}$ is maximal. To see this, we generalize a result of Bongartz ([B] and [H]):

Lemma 4.26. *Let R be a ring and A and B be modules. Assume $Ext_R(B, B^{(\kappa)}) = 0$ for all cardinals κ. Then there are a cardinal λ and a module C such that $Ext_R(B, C) = 0$ and there is an exact sequence*

$$0 \to A \to C \to B^{(\lambda)} \to 0.$$

Proof. Take a cardinal λ and extensions

$$(*) \qquad 0 \to A \to E_\alpha \to B \to 0, \quad \alpha < \lambda,$$

so that these extensions generate the group $Ext_R(B, A)$. Let

$$(**) \qquad 0 \to A \to C \xrightarrow{\pi} B^{(\lambda)} \to 0$$

be the extension obtained by pushing out the direct sum extension

$$0 \to A^{(\lambda)} \to \oplus \sum_{\alpha<\lambda} E_\alpha \to B^{(\lambda)} \to 0$$

along $\phi \in Hom_R(A^{(\lambda)}, A)$ defined by $\phi((a_\alpha; \alpha < \lambda)) = \sum_{\alpha<\lambda} a_\alpha$. Consider the long exact sequence

$$0 \to Hom_R(B, A) \to Hom_R(B, C) \to Hom_R(B, B^{(\lambda)}) \xrightarrow{\delta} Ext_R(B, A) \to$$
$$\to Ext_R(B, C) \xrightarrow{Ext_R(B,\pi)} Ext_R(B, B^{(\lambda)}) = 0 \to \dots$$

induced by (**) and by the functor $Hom_R(B, -)$. Since the extensions (*) generate $Ext_R(B, A)$, the connecting $\mathbb{Z}$-homomorphism δ is onto. Hence, the $\mathbb{Z}$-homomorphism $Ext_R(B, \pi)$ is a monomorphism. This proves that $Ext_R(B, C) = 0$. $\square$

Lemma 4.27. *Let R be a right noetherian ring such that R is not right hereditary. Assume $\mathcal{IT}$ is κ-saturated, for all $\kappa < \aleph_0$ and all $\kappa \leq gen(M)$, M being any indecomposable injective module. Then R is a QF-ring.*

Proof. Let B be an indecomposable injective module. Since R is right noetherian, we have $Ext_R(B, B^{(\kappa)}) = 0$ for all cardinals κ. Let A be a non-injective module. By 4.26, there are a cardinal λ and a module C such that $Ext_R(B, C) = 0$ and there is an extension

$$(***) \qquad 0 \to A \to C \to B^{(\lambda)} \to 0.$$

Assume C is injective. Since R is right noetherian, but not right hereditary, there is a non-projective finitely generated right ideal I of R. Applying the functor $Hom_R(R/I, -)$ to the exact sequence (***), we get

$$0 = Ext_R(R/I, B^{(\lambda)}) \to Ext^2_R(R/I, A) \to Ext^2_R(R/I, C) = 0,$$

whence $Ext^2_R(R/I, A) = 0$. Applying the functor $Hom_R(-, A)$ to the exact sequence $0 \to I \to R \to R/I \to 0$, we get

$$0 = Ext_R(R, A) \to Ext_R(I, A) \to Ext^2_R(R/I, A) = 0.$$

Since $gen(I) < \aleph_0$, A is injective, a contradiction.
Thus, C is not injective. Since $Ext_R(B, C) = 0$, our premise implies that B is projective.
This proves that any injective module is projective, and R is a QF-ring by a theorem of Faith and Walker ([AF]). □

Theorem 4.28. *Let R be a ring such that $\mathcal{IT}$ is fully saturated. Then $R = R'(\boxplus R'')$, where R'' is a semisimple ring and R' is an indecomposable ring such that each non-projective right R'-module is i-test. Moreover, either*
(Ia) R' is isomorphic to a full matrix ring over a local QF-ring, or
(Ib) R' is a simple von Neumann regular ring such that all right ideals are countably generated and all simple right R'-modules are isomorphic, or
(II) R' is Morita equivalent to $UT_2(S)$ for a skew-field S.

Proof. By 4.22 and 4.27. □

So far, all results of this section were proved in ZFC. We finish by showing that 4.28 can be improved in the models of ZFC satisfying the Shelah's uniformization principle UP (see 2.3):

Lemma 4.29. *Let κ be a cardinal such that* $\mathrm{cf}\,(\kappa) = \aleph_0$. *Assume* UP_κ. *Let R be a non-right perfect ring such that $card(R) < \kappa$. Then $\mathcal{PT}$ is not 1-saturated, and $\mathcal{IT}$ is not κ^+-saturated.*

Proof. Since R is not semisimple, [Os, Corollary 2.23] implies that there exists a cyclic non-injective module, N. Clearly, $card(N) \le card(R) < \kappa$. By 2.2 and 2.4, there is a non-projective module M such that $gen(M) \le \kappa^+$ and $Ext_R(M, N) = 0$. Hence, N is not p-test, and M is not i-test. □

Theorem 4.30. *Assume UP. Let R be a ring such that $\mathcal{IT}$ is fully saturated. Then $R = R'(\boxplus R'')$, where R'' is a semisimple ring and either*
(I) R' is isomorphic to a full matrix ring over a local QF-ring, or
(II) R' is Morita equivalent to $UT_2(S)$ for a skew-field S.

Proof. By 4.28 and 4.29. □

Corollary 4.31. *Assume UP. Let R be a right non-singular ring. Then the following conditions are equivalent:*
(i) $\mathcal{IT}$ is fully saturated,
(ii) $\mathcal{PT}$ is fully saturated,
(iii) $R = R'(\boxplus R'')$, where R'' is a semisimple ring and there is a skew-field S such that R' is Morita equivalent to $UT_2(S)$.

§5 Applications: a solution to Menini's problem

In the present section, we apply results of §4 to solve a problem (due to C.Menini) concerning representable equivalences of module categories. First, we recall the categorial background of the problem:

There are several important generalizations of the celebrated Morita theorem concerning equivalence of module categories. In most of them, there is a representing module inducing an equivalence of module subcategories. In this way, the notion of a quasi-progenerator was introduced by K.R.Fuller ([F]). Also the (general) tilting modules appear in this context, as shown in [CbF].

There is a class of modules comprising both quasi-progenerators and tilting modules: A module $P \in Mod\text{-}R$ is a *$*$-module* provided P induces (via $Hom_R(P,-)$ and $-\otimes_{R'} P$) an equivalence between $Gen(P_R)$ and $Cog(P^*_{R'})$, where $R' = End(P_R)$ and $P^* = Hom_R(P,Q)$ for an injective cogenerator $Q \in Mod\text{-}R$. Here, $Gen(M_R)$ and $Cog(M_R)$ denote the category of all modules generated and cogenerated, respectively, by M.

The study of $*$-modules is motivated by the following representation theorem of Menini and Orsatti ([MeO]): if B and C are equivalent categories, where $B \subseteq Mod\text{-}R'$ is such that $R' \in B$ and B is closed under submodules, and $C \subseteq Mod\text{-}R$ is closed under direct sums and factors, then there is a unique $*$-module P such that $C = Gen(P_R)$, $B = Cog(P^*_{R'})$, and P induces the equivalence between B and C. The class of all $*$-modules is denoted by $\mathcal{STAR}$.

So far, there is no explicit description of the class $\mathcal{STAR}$ over an arbitrary associative ring with unit. Nevertheless, by a result of the author, all $*$-modules are finitely generated ([T2]). Moreover, there is a general criterion due to R.Colpi: a finitely generated module P is in $\mathcal{STAR}$ if and only if P satisfies the condition $C(\kappa)$ for all cardinals κ. Here, $C(\kappa)$ denotes the *Colpi's condition* for κ, i.e. the assertion:
"for every submodule M of $P^{(\kappa)}$, the condition $M \in Gen(P_R)$ is equivalent to the injectivity of the canonical group homomorphism $Ext_R(P,M) \to Ext_R(P,P^{(\kappa)})$".
Clearly, $C(0)$ always holds, and it is easy to see that $C(\kappa)$ implies $C(\kappa')$ for all $\kappa' \leq \kappa$.

If $\lambda > 0$ is a cardinal and P a module, then P is said to be a *$*_\lambda$-module* provided P is finitely generated and P satisfies $C(\kappa)$ for all $\kappa < \lambda$. The class of all $*_\lambda$-modules is denoted by $\mathcal{S}_\lambda$. In particular $\mathcal{S}_1 = \mathcal{FG}$, the class of all finitely generated modules. Moreover, $*_{\aleph_0}$-modules are called *almost $*$-modules.* The class of all almost $*$-modules is denoted by $\mathcal{ASTAR}$. Thus, we have the following decreasing chain of subclasses of $Mod\text{-}R$:

$$(\clubsuit)\qquad \begin{aligned} \mathcal{FG} = \mathcal{S}_1 \supseteq \mathcal{S}_2 \supseteq \dots \supseteq \mathcal{ASTAR} = \mathcal{S}_{\aleph_0} \supseteq \mathcal{S}_{\aleph_1} \supseteq \dots \\ \dots \supseteq \mathcal{S}_\lambda \supseteq \mathcal{S}_{\lambda^+} \supseteq \dots \supseteq \cap_\lambda \mathcal{S}_\lambda = \mathcal{STAR}, \end{aligned}$$

where $\mathcal{S}_\lambda = \cap_{\kappa<\lambda}\mathcal{S}_\kappa$ provided $\lambda = \aleph_\alpha$ and either $\alpha = 0$ or α is a limit ordinal.

A possible approach to $*$-modules is by studying the class $\mathcal{STAR}$ "from above", i.e. by means of investigations of the particular classes $\mathcal{S}_\lambda$ and the categorical equivalences induced by their elements. An important feature of each $*_\lambda$-module (discovered by R.Colpi) is that it induces an equivalence between certain natural subcategories of $Gen(P_R)$ and $Cog(P^*_S)$, depending on λ.

Of course, a question arises whether the hierarchy of the classes $\mathcal{S}_\lambda$ can be simplified, i.e. whether the inclusions of S_{λ^+} into S_λ in ($\clubsuit$) are strict. An essential simplification would follow from a positive solution to the following problem of C.Menini:

"Does $\mathcal{STAR} = \mathcal{ASTAR}$ hold (for an arbitrary ring) ?"

Nevertheless, the answer to this question is negative: we shall show that for each infinite cardinal λ there exist a hereditary artinian ring R and a 2-generated $*_\lambda$-module P such that $C(\lambda)$ does not hold. Hence, in this case, the inclusion of S_{λ^+} into S_λ in (♣) is strict. Moreover, this means that *for each $\lambda \geq \aleph_0$, the Colpi's condition $C(\lambda)$ is independent of "$C(\kappa)$ for all $\kappa < \lambda$".*

Our construction of P is based on Cohn-Schofield solution of Artin's problem for skew-field extensions: We start with a quadratic extension S of a skew field T constructed by Cohn and such that $dim_T S = \lambda$. Then we take the ring $R = UT(1, S, T)$ (see 4.12 - 4.14) and prove that the module $P = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}; a, b \in S\} \subset M_2(S)$ is the suitable one. The first step of the proof uses a transfer lemma which expresses the vanishing of $Ext_R(P, -)$ in terms of a non-commutative linear algebra assertion over T. Then $C(\kappa)$ is shown to hold for all $\kappa < \lambda$ provided the right dimension of T over a certain sub-skew-field is λ. The final step consists in an analysis of the original Cohn's construction.

First, we fix our notation for the rest of this section:

Definition 5.1. Let T and S be skew-fields such that S is a quadratic extension of T (i.e. $T \subset S$ and $dim S_T = 2$). Fix an element $x \in S \setminus T$. Moreover, put $R = UT(1, S, T)$, i.e. let R be the subring of $M_2(S)$ consisting of all matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $a \in T$ and $b, c \in S$. Further, denote by P the module $\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}; a, b \in S\} \subset M_2(S)$.

Lemma 5.2. *For each $t \in T$ there are unique $\phi(t), D(t) \in T$ such that*

$$tx = x\phi(t) + D(t).$$

The mapping $\phi : T \to T$ is an injective ring homomorphism, $D : T \to T$ is a ϕ-differentiation of T, and $dim_{\phi(T)} T = dim_T S - 1$.

Proof. Well-known. □

Lemma 5.3. *R is left and right artinian, and left and right hereditary. Upto isomorphism, the modules P and $R/Soc(R)$ are the only indecomposable injective modules in Mod-R, the first one being 2-generated and the second being simple. For each $M \in Mod$-R there are a decomposition $M \simeq D(M) \oplus R(M)$ and a unique cardinal κ_M such that $D(M)$ is the trace of $R/Soc(R)$ in M and*

$$Soc(P^{(\kappa_M)}) = Soc(R(M)) \trianglelefteq R(M) \trianglelefteq P^{(\kappa_M)}.$$

Proof. Easy, using 4.13. □

The following lemma transfers vanishing of $Ext_R(P, -)$ into a linear algebraic assertion over T:

Lemma 5.4. *Define $Y_M = \{(a_\alpha)_{\alpha<\kappa_M} \in S^{(\kappa_M)}; \begin{pmatrix} a_\alpha & 0 \\ 0 & 0 \end{pmatrix}_{\alpha<\kappa_M} \in R(M)\}$ for each $M \in Mod$-R. Then Y_M is a right T-submodule of $S^{(\kappa_M)}$. Moreover, the following conditions are equivalent:*
(i) $Ext_R(P, M) = 0$,
(ii) $Y_M + Y_M.x = S^{(\kappa_M)}$ (as abelian groups).

Proof. Since $R(M)$ is a submodule of $P^{(\kappa_M)}$, we see that Y_M is a right T-submodule of $S^{(\kappa_M)}$. In view of 5.3, w.l.o.g. we can assume that $D(M)=0$, i.e. $Soc(P^{(\kappa)}) = Soc(M) \trianglelefteq M \trianglelefteq P^{(\kappa)}$, where $\kappa=\kappa_M$. Put $p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in P$, $p_1 = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in P$, $q = (\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}) \in R^{(2)}$, $q_0 = (\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) \in R^{(2)}$ and $q_1 = (\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) \in R^{(2)}$. Clearly, $P = p_0R + p_1R$ and $P \simeq R^{(2)}/(qR \oplus q_0R \oplus q_1R)$. Since $Ann_R(q) = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}; a \in T, b \in S\}$, the elements of $Hom_R(qR, M)$ are in one-one correspondence with the elements of $Soc(M) = Soc(P^{(\kappa)}) = \{\begin{pmatrix} 0 & b_\alpha \\ 0 & 0 \end{pmatrix}_{\alpha<\kappa}; (b_\alpha)_{\alpha<\kappa} \in S^{(\kappa)}\}$. Similarly, $Hom_R(q_0R \oplus q_1R, M) \simeq (Soc(M))^{(2)}$. Of course, $Hom_R(R^{(2)}, M) \simeq M^{(2)}$. Then $Ext_R(P,M)=0$ iff the canonical homomorphism $Hom_R(R^{(2)}, M) \to Hom_R(qR, M)$ induced by restriction is onto iff for each $\begin{pmatrix} 0 & b_\alpha \\ 0 & 0 \end{pmatrix}_{\alpha<\kappa}$ with $(b_\alpha)_{\alpha<\kappa} \in S^{(\kappa)}$ and all $\begin{pmatrix} 0 & c^i_\alpha \\ 0 & 0 \end{pmatrix}_{\alpha<\kappa}$ with $(c^i_\alpha)_{\alpha<\kappa} \in S^{(\kappa)}$, $i=0,1$, there are some $\begin{pmatrix} a^i_\alpha & b^i_\alpha \\ 0 & 0 \end{pmatrix}_{\alpha<\kappa}$, with $(a^i_\alpha)_{\alpha<\kappa} \in Y_M$ and $(b^i_\alpha)_{\alpha<\kappa} \in S^{(\kappa)}$, $i=0,1$, and such that $b^i_\alpha = c^i_\alpha$ for $i=0,1$ and $a^0_\alpha.x - a^1_\alpha = b_\alpha$, for all $\alpha<\kappa$. The last assertion is equivalent to $Y_M + Y_M.x = S^{(\kappa)}$. □

Note that for any cardinal $\kappa > 0$ and any right T-submodule Y of $S^{(\kappa)}$ there is some $M \in Mod\text{-}R$ such that $Y = Y_M$ and $\kappa = \kappa_M$. Moreover, M is injective iff $Y_M = S^{(\kappa_M)}$.

Lemma 5.5. *Let $\kappa > 0$ be a cardinal and M a submodule of $P^{(\kappa)}$. Then $\kappa_M \leq \kappa$ and*
(i) $M \in Gen(P_R)$ iff there is a cardinal $\gamma \leq \kappa$ such that $M \simeq P^{(\gamma)}$ is a summand of $P^{(\kappa)}$,
(ii) the canonical group homomorphism $Ext_R(P,M) \to Ext_R(P,P^{(\kappa)})$ is injective iff $Y_M + Y_M.x = S^{(\kappa_M)}$ (as abelian groups).

Proof. (i) By 5.3, if $M \in Gen(P_R)$, then M is an injective submodule of $P^{(\kappa)}$. Hence, $M \simeq P^{(\gamma)}$ for some $\gamma \leq \kappa$.
(ii) By 5.3, $Ext_R(P, P^{(\kappa)}) = 0$, and the result follows from 5.4. □

Lemma 5.6. *Put $\delta = dim T_{\phi(T)}$.*
(i) If κ is a cardinal such that $\delta \leq \kappa$, then $C(\kappa)$ does not hold.
(ii) If $\delta \geq \aleph_0$, then $C(\kappa)$ holds for all $\kappa < \delta$.
(iii) If $\delta < \aleph_0$ and $n < \aleph_0$ is such that $2n \leq \delta$, then $C(n)$ holds.

Proof. Let κ be a cardinal. By 5.3 and 5.5, $C(\kappa)$ is equivalent to the assertion "$Y + Yx \neq S^{(\kappa)}$ for all proper right T-submodules Y of $S^{(\kappa)}$". For each $\alpha < \kappa$, let $\pi_\alpha : S^{(\kappa)} \to S$ be the α-th projection. Let $B = \{1_\alpha; \alpha < \kappa\}$ be the canonical basis of the right S-module $S^{(\kappa)}$. For $\alpha < \kappa$ let $x_\alpha = 1_\alpha x$. Clearly, $B \cup \{x_\alpha; \alpha < \kappa\}$ is a basis of the right T-module $S^{(\kappa)}$.

(i): Assume $\delta \leq \kappa$. Let $\{t_\nu; \nu < \delta\}$ be a right $\phi(T)$-basis of T. Let Y be a right T-submodule of $S^{(\kappa)}$ generated by $B \cup \{y_\alpha; \alpha < \kappa\}$, where $\pi_0(y_\nu) = xt_\nu$ for all $\nu < \delta$, $\pi_{\alpha+1}(y_\alpha) = x$ for all $\alpha < \kappa$, and $\pi_\beta(y_\alpha) = 0$ otherwise. Since $x_0 \notin Y$, Y is a proper submodule of $S^{(\kappa)}$. By 5.2, $T \oplus Tx = T \oplus x.\phi(T)$, whence $\oplus \sum_{\alpha<\kappa} x_\alpha \phi(T) \subseteq Y + Yx$. Thus,

$$x_0 T = \oplus \sum_{\nu<\delta} x_0 t_\nu \phi(T) \subseteq Y + Yx,$$

and $Y + Yx = S^{(\kappa)}$. Therefore, $C(\kappa)$ does not hold.

(ii) and (iii): Take $0 < \kappa < \delta$ such that $2\kappa \leq \delta$ provided $\delta < \aleph_0$. Proving indirectly, assume that $C(\kappa)$ does not hold. Then there is a maximal right T-submodule Y of $S^{(\kappa)}$ such that $Y + Yx = S^{(\kappa)}$. Let $D = \{d_\alpha; \alpha < \lambda\}$ be a right T-basis of Y. Since $dimS_T = 2$, we have $\lambda = 2\kappa - 1$ provided κ is finite, and $\lambda = \kappa$ otherwise. By 5.2, $T \oplus Tx = T \oplus x.\phi(T)$, whence

$$S^{(\kappa)} = Y + Yx = \sum_{\alpha<\lambda} d_\alpha(T \oplus Tx) = \sum_{\alpha<\lambda} d_\alpha(T \oplus x.\phi(T)) = Y + \sum_{\alpha<\lambda} d_\alpha.x.\phi(T).$$

In particular, $dim(S^{(\kappa)}/Y)_{\phi(T)} = dimT_{\phi(T)} = \delta \leq \lambda$, a contradiction. □

The following theorem provides a negative answer to Menini's question:

Theorem 5.7. *Let λ be an infinite cardinal and K, L be the skew-fields constructed by Cohn for $\alpha = \lambda$ and $\beta = 2$. Then L is a quadratic extension of K and $dim_K L = \lambda$. Put $S = L$, $T = K$, and let $P = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}; a, b \in S\}$. Then P is a 2-generated module, $C(\kappa)$ holds for all $\kappa < \lambda$, but $C(\kappa')$ does not hold for any $\kappa' \geq \lambda$.*

Proof. By 5.3, P is 2-generated. In view of 5.6(i) and (ii), it suffices to show that $dimT_{\phi(T)} = \lambda$. To this aim, we analyze the construction of Cohn ([Co, pp.124-126]), using partly the notation thereof: We have $\phi(K) \subseteq E_\mu(t)$ and, by [Co, pp.125-126], $x_{\mu 11} \notin E_\mu(t)$, for all $\mu < \alpha$. Since $x_{\nu 11} \in E_\mu(t)$ for all $\nu \neq \mu < \alpha$, the set $\{x_{\mu 11}; \mu < \alpha\}$ is not only a left, but also a right $\phi(K)$-independent subset of K. Since $card(K) = card(L) = \alpha$, the assertion follows. □

Theorem 5.7 implies that, in general, $*$-modules cannot easily be reached "from above", using $*_\lambda$-modules and their hierarchy (♣). On the other hand, there are many particular cases in which $*$-modules can be reached "from below", using quasi-progenerators or tilting modules. For example, $*$-modules over commutative rings are exactly the quasi-progenerators (by [CMe] and [T2]). $*$-modules over finite dimensional algebras coincide with those modules that are tilting modulo their annihilators (a result of [CMe] and [DH]).

Open Problems

(1) In 1.6, we have proved *in ZFC* that $\mathcal{PT}$ is a proper class for any right perfect ring R. If R is not right perfect, then 2.5 shows that it is *consistent with ZFC* that $\mathcal{PT}$ is empty. On the other hand, if R is right hereditary, then it is *consistent with ZFC* that $\mathcal{PT}$ is a proper class (see 3.13).
What is the possible size of $\mathcal{PT}$ in the case when R is non-right perfect and non-right hereditary ? Is then the assertion of 2.5 a theorem of ZFC ? The problem is open even in the very particular case of full endomorphism rings of infinite dimensional linear spaces over skew-fields (cp. 2.6 and 3.20).

(2) By 4.14, all rings of type (II) are Morita equivalent to some $UT(m, S, T)$. Moreover, 4.15 provides a description using block upper triangular matrix rings. Which of these matrix rings really are of type (II) ? For partial answers, see 4.17-19.

(3) By 4.28, each ring R of type (Ia) such that $\mathcal{IT}$ is maximal is isomorphic to a full matrix ring over a local QF-ring R'. Is R' actually an artinian valuation ring (as in 4.5) ?

(4) By 5.7, for each $\lambda \geq \aleph_0$, there is an artinian hereditary ring R such that $S_{\lambda^+} \subset S_\lambda$. Is the same true for each $0 < \lambda < \aleph_0$? In particular, can the methods of §5 be used also in this case, replacing the Cohn's quadratic extensions $T \subset S$ by the ones constructed by Schofield ([Sc]) ?

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