COVERS, ENVELOPES, AND COTORSION THEORIES

Lecture notes for the workshop
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Introduction

Module theory provides a general framework for the study of linear representations of various mathematical objects. For example, given a field K, representations of a quiver Q may be viewed as modules over the path algebra $\langle KQ \rangle$. Similarly, representations of a group G coincide with modules over the group algebra KG; representations of a Lie algebra L are modules over the universal enveloping algebra $U(L)$ etc.

In general, there is little hope to describe all modules over a given ring, or algebra, R. Unless R is of finite representation type, that is, unless each module is a direct sum of indecomposable ones, we have to restrict our study to particular classes of modules. Once we understand the structure of a class, $\mathcal{C}$, we may try to approximate arbitrary modules by the modules from $\mathcal{C}$.

Since the early 1960's, this approach has successfully been used to investigate injective envelopes, projective covers as well as pure-injective envelopes of modules, [2], [48]. An independent research of Auslander, Reiten and Smalø in the finite dimensional case, and Enochs and Xu for arbitrary modules, has led to a general theory of left and right approximations – or preenvelopes and precovers – of modules, [7], [8], [26] and [50].

The notions of a preenvelope and a precover are dual in the category theoretic sense. In the late 1970's, Salce noticed that these notions are also tied up by a homological notion of a complete cotorsion theory [42]. The point is that though there is no duality between the categories of all modules, complete cotorsion theories make it possible to produce preenvelopes once we know precovers exist and vice versa.

In a recent work, Eklof and the author proved that complete cotorsion theories are abundant. For example, any cotorsion theory cogenerated by a set of modules is complete, [22]. Consequently, many classical cotorsion theories are complete. In this way, Enochs proved that the flat cotorsion theory is complete, thus proving the celebrated Flat Cover Conjecture (FCC): every module over any ring has a flat cover [9]. Similarly, the author proved that all modules have torsion-free covers [45].

In the finite dimensional case, Auslander and Reiten studied approximations of modules induced by tilting and cotilting modules [7]. This theory has recently been extended to arbitrary modules [4], [5], [19]. For example, it turned out that a torsion class of modules $\mathcal{T}$ provides for special preenvelopes if and only if $\mathcal{T}$ is generated by a tilting module. Here, tilting modules are allowed to be infinitely generated: in fact, all non-trivial examples for $R = \mathbb{Z}$ (or when R is a small Dedekind domain) are infinitely generated, [23], [32], [47].

The lecture notes are divided into four chapters accompanied by a list of open problems and references.

In Chapter 1, we present basics of the general theory of approximations and cotorsion theories of modules as developed by Enochs and Salce.

In Chapter 2, we prove that complete cotorsion theories are abundant, following the recent works of Eklof and the author, [22], [23].

Chapter 3 consists of applications. We construct various particular approximations of modules over arbitrary rings. We prove the FCC, as well as existence of torsion-free covers and of cotorsion envelopes in the sense of Enochs, Warfield and Matlis. We also construct approximations by modules of finite homological dimensions in the spirit of [1].

In Chapter 4, we relate the approximation theory to tilting and cotilting theory of (infinitely generated) modules, following recent works of Angeleri-Hügel, Colpi, Tonolo and the author, [5], [19] and [45].

Some of the results presented in the lecture notes can be extended to more general categories - in particular, to Grothendieck categories, cf. [14], [24], [27], [34], [41], [46]. Nevertheless, our setting is that of modules over associative unital rings. Namely, together with developing the general approximation theory we aim at applications to the structure of particular classes of modules. Covers and envelopes are unique, so they provide for invariants of modules similar to the Bass numbers or dual Bass numbers in the sense of [50, Chap. 5]. It is the study of these invariants that appears to be one of the challenging tasks for future research in module theory.

The importance of each of the numerous envelopes, covers and cotorsion theories depends very much on the ring in case. We will illustrate this throughout the text in the case of domains, in particular the Prüfer and the Dedekind ones. For example, if R is a Prüfer domain then the complete cotorsion theory $(\mathcal{P}_1, \mathcal{D}I)$ plays an important role: several classical results of Fuchs and Salce [29] can be proved, and generalized, by applying the approximation theory to this case. This will be shown in Chapter 4.

Salce's result on envelopes and covers induced by complete cotorsion theories provides also for a new insight in some of the classical results. For example, Enochs' theorem on the existence of torsion-free covers of modules over domains [25] implies the existence of Warfield's cotorsion hulls [29, XII.4] and vice versa, a fact by no means evident from the classical proofs.

R will always denote an (associative unital) ring. For a ring R, we denote by Mod-R the category of all (unitary right R-) modules. Let S be a commutative ring such that R is an S-algebra. Let E be an injective cogenerator of Mod-S and N be a left R-module. Then N is an R, S-bimodule. Put $N^* = \mathrm{Hom}_S(N, E)$. If $M \cong N^*$ as S, R-bimodules then M is called a *dual module* (of N). Similarly, we define the R, S-bimodule N^{**}.

In the particular case when $S = \mathbb{Z}$, $E = \mathbb{Q}/\mathbb{Z}$, the dual module $N^* = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$ is called the *character module* of N and denoted by N^c. If R is a k-algebra over a field k then any finite k-dimensional module M is dual (since $M \cong M^{**}$ where $S = E = k$). If R is commutative then the choice of $S = R$ and E an injective cogenerator of Mod-R provides for another instance of a dual module.

For a module M denote by $\mathrm{Gen}(M)$ the class of all modules generated by M, that is, of all homomorphic images of arbitrary direct sums of copies of M. Let $\mathrm{Pres}(M)$ be the class of all modules presented by M, so $\mathrm{Pres}(M) = \{N \in \text{Mod-}R \mid \exists K \in \mathrm{Gen}(M)\, \exists \kappa : N \cong M^{(\kappa)}/K\}$. Denote by $\mathrm{Add}(M)$ the class of all direct summands of arbitrary direct sums of copies of M.

Dually, we define the classes of all modules cogenerated and copresented by M, $\mathrm{Cogen}(M)$ and $\mathrm{Copres}(M)$, and the class $\mathrm{Prod}(M)$ of all direct summands of arbitrary direct products of copies of M.

Denote by Q the maximal quotient ring of R. If R is a (commutative integral) domain then Q coincides with the quotient field of R.

Let R be a ring, I be a right ideal of R, and M be a module. Then M is *I-divisible* provided that $\mathrm{Ext}^1_R(R/I, M) = 0$. If $I = rR$ for some $r \in R$ then the term *r-divisible* will also be used to denote I-divisibility. Following Lam [38, 3.16], we call M *divisible* if M is r-divisible for all $r \in R$. Denote by $\mathcal{DI}$ the class of all divisible modules.

Let I be a left ideal of R. A module M is *I-torsion-free* provided that $\mathrm{Tor}_1^R(M, R/I) = 0$. If $I = Rr$ for some $r \in R$ then the term *r-torsion-free* will also be used to denote I-torsion-freeness. M is *torsion-free* if M is r-torsion-free for all $r \in R$. The class of all torsion-free modules is denoted by $\mathcal{TF}$.

If $r \in R$ with $\mathrm{Ann}_r(r) = 0$ then M is r-divisible iff $Mr = M$. Similarly, if $\mathrm{Ann}_l(r) = 0$ then M is r-torsion-free iff $mr = 0$ implies $m = 0$ for all $m \in M$. In particular, these characterizations hold true for any non-zero $r \in R$ in the case when R is a domain.

Denote by $\mathcal{P}_n$ ($\mathcal{I}_n$) the class of all modules of projective (injective) dimension $\leq n$, and by $\mathcal{FL}$ the class of all flat modules. Then $\mathcal{P}_0 \subseteq \mathcal{FL} \subseteq \mathcal{TF}$ and $\mathcal{I}_0 \subseteq \mathcal{DI}$ for any ring R.

A submodule A of a module B is *pure* ($A \subseteq_* B$, for short) if for each finitely presented module F, the functor $\mathrm{Hom}(F, -)$ preserves exactness of the sequence $0 \to A \to B \to B/A \to 0$. Modules that are injective with respect to pure embeddings are called *pure-injective*. The class of all pure-injective modules is denoted by $\mathcal{PI}$. For example, any dual module is pure injective. In fact, a module M is pure-injective iff M is a direct summand in a dual module, cf. [36], [50].

A ring R is *right coherent* provided that each finitely generated right ideal is finitely presented. For example, any Prüfer domain is right coherent.

For further preliminaries on general rings and modules, we refer to [2], and for modules over domains to [29]. We will also need several well-known facts from homological algebra: the existence of the long exact sequence for Ext^n_R, the fact that $\mathrm{Ext}^1_R(M, N) = 0$ iff each extension of N by M splits, and the Ext^1 - Tor_1 relations of [12, VI.5].

1 Approximations of modules

We start with introducing the basic notions of the approximation theory in the setting of module categories.

Definition 1.1 Let M be a module and $\mathcal{C}$ be a class of modules closed under isomorphic images and direct summands.

A map $f \in \operatorname{Hom}_R(M, C)$ with $C \in \mathcal{C}$ is called a $\mathcal{C}$*-preenvelope* of M provided that the abelian group homomorphism $\operatorname{Hom}_R(f, C')$ is surjective for each $C' \in \mathcal{C}$. That is, for each homomorphism $f' : M \to C'$ there is a homomorphism $g : C \to C'$ such that $f' = gf$:

$$\begin{array}{ccc} M & \xrightarrow{f} & C \\ \| & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f'} & C' \end{array}$$

The $\mathcal{C}$-preenvelope f is a $\mathcal{C}$*-envelope* of M if g is an automorphism whenever $g \in \operatorname{Hom}_R(C, C)$ and $f = gf$.

Example 1.2 For a module M, denote by $E(M)$ and $PE(M)$ the injective and pure-injective hulls of M. Then the embedding $M \hookrightarrow E(M)$ is an $\mathcal{I}_0$-envelope of M, and $M \hookrightarrow PE(M)$ is a $\mathcal{PI}$-envelope of M.

Clearly, a $\mathcal{C}$*-envelope* of M is unique in the following sense: if $f : M \to C$ and $f' : M \to C'$ are $\mathcal{C}$*-envelopes* of M then there is an isomorphism $g : C \to C'$ such that $f' = gf$.

In general, a module M may have many non-isomorphic $\mathcal{C}$-preenvelopes, but no $\mathcal{C}$-envelope (see Chapter 4). Nevertheless, if the $\mathcal{C}$-envelope exists, it is recognized as the minimal $\mathcal{C}$-preenvelope in the sense of the following lemma:

Lemma 1.3 *Let $f : M \to C$ be a $\mathcal{C}$-envelope and $f' : M \to C'$ a $\mathcal{C}$-preenvelope of a module M. Then*

1. *$C' = D \oplus D'$, where $\operatorname{Im} f' \subseteq D$ and $f' : M \to D$ is a $\mathcal{C}$-envelope of M;*
2. *f' is a $\mathcal{C}$-envelope of M iff C' has no proper direct summands containing $\operatorname{Im} f'$.*

Proof. 1. By definition, there are homomorphisms $g : C \to C'$ and $g' : C' \to C$ such that $f' = gf$ and $g'g$ is an automorphism of C. So $D = \operatorname{Im} g \cong C$ is a direct summand in C' containing $\operatorname{Im} f'$, and the assertion follows.

2. By part 1. ∎

Definition 1.4 A class $\mathcal{C} \subseteq$ Mod-R is a *preenvelope class* (*envelope class*) provided that each module has a $\mathcal{C}$-preenvelope ($\mathcal{C}$-envelope).

For example, the classes $\mathcal{I}_0$ and $\mathcal{PI}$ from Example 1.2 are envelope classes of modules.

Now, we briefly discuss the dual concepts:

Definition 1.5 Let $\mathcal{C} \subseteq$ Mod-R be closed under isomorphic images and direct summands. Let $M \in$ Mod-R. Then $f \in \mathrm{Hom}_R(C, M)$ with $C \in \mathcal{C}$ is a *$\mathcal{C}$-precover* of M provided that the abelian group homomorphism $\mathrm{Hom}_R(C', f) : \mathrm{Hom}_R(C', C) \to \mathrm{Hom}_R(C', M)$ is surjective for each $C' \in \mathcal{C}$.

A $\mathcal{C}$-precover $f \in \mathrm{Hom}_R(C, M)$ of M is called a *$\mathcal{C}$-cover* of M provided that $fg = f$ and $g \in \mathrm{End}(C_R)$ implies that g is an automorphism of C.

$\mathcal{C} \subseteq$ Mod-R is a *precover class* (*cover class*) provided that each module has a $\mathcal{C}$-precover ($\mathcal{C}$-cover). [1]

Example 1.6 Each module M has a $\mathcal{P}_0$-precover. Moreover, M has a $\mathcal{P}_0$-cover iff M has a projective cover in the sense of Bass [2, §26]. So $\mathcal{P}_0$ is always a precover class, and it is a cover class iff R is a right perfect ring.

$\mathcal{C}$-covers may not exist in general, but if they exist, they are unique up to isomorphism. As in Lemma 1.3, we get

Lemma 1.7 *Let $f : C \to M$ be the $\mathcal{C}$-cover of M. Let $f' : C' \to M$ be any $\mathcal{C}$-precover of M. Then*

1. *$C' = D \oplus D'$, where $D \subseteq \mathrm{Ker}\, f'$ and $f' \restriction D'$ is a $\mathcal{C}$-cover of M;*
2. *f' is a $\mathcal{C}$-cover of M iff C' has no non-zero direct summands contained in $\mathrm{Ker}\, f'$.*

Proof. Dual to the proof of Lemma 1.3. ■

Wakamatsu proved that under rather weak assuptions on the class $\mathcal{C}$, $\mathcal{C}$-envelopes and $\mathcal{C}$-covers are special in the sense of the following definition:

Definition 1.8 Let $\mathcal{C} \subseteq$ Mod-R. Define

$$\mathcal{C}^{\perp} = \mathrm{Ker}\,\mathrm{Ext}^1_R(\mathcal{C}, -) = \{N \in \text{Mod-}R \mid \mathrm{Ext}^1_R(C, N) = 0 \text{ for all } C \in \mathcal{C}\}$$

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[1] $\mathcal{C}$-preenvelopes and $\mathcal{C}$-precovers are sometimes referred to as left and right $\mathcal{C}$-approximations; preenvelope and precover classes are then called covariantly finite and contravariantly finite, respectively, cf. [7] and [8].

$${}^{\perp}\mathcal{C} = \operatorname{Ker}\operatorname{Ext}^1_R(-,\mathcal{C}) = \{N \in \text{Mod-}R \mid \operatorname{Ext}^1_R(N,C) = 0 \text{ for all } C \in \mathcal{C}\}.$$

If $\mathcal{C} = \{C\}$ then we simply write $C^{\perp}$ and ${}^{\perp}C$.

Let $M \in \text{Mod-}R$. A $\mathcal{C}$-preenvelope $f : M \to C$ of M is called *special* provided that f is injective and $\operatorname{Coker} f \in {}^{\perp}\mathcal{C}$. In other words, there is an exact sequence

$$0 \longrightarrow M \xrightarrow{f} C \to D \longrightarrow 0$$

with $C \in \mathcal{C}$ and $D \in {}^{\perp}\mathcal{C}$.

Dually, a $\mathcal{C}$-precover $f : C \to M$ of M is called *special* if f is surjective and $\operatorname{Ker} f \in \mathcal{C}^{\perp}$.

Lemma 1.9 *Let M be a module. Let $\mathcal{C}$ be a class of modules closed under extensions.*

1. *Assume $\mathcal{I}_0 \subseteq \mathcal{C}$. Let $f : M \to C$ be a $\mathcal{C}$-envelope of M. Then f is special.*
2. *Assume $\mathcal{P}_0 \subseteq \mathcal{C}$. Let $f : C \to M$ be a $\mathcal{C}$-cover of M. Then f is special.*

Proof. 1. By assumption, $M \hookrightarrow E(M)$ factors through f, hence f is injective. So there is an exact sequence

$$0 \longrightarrow M \xrightarrow{f} C \xrightarrow{g} D \longrightarrow 0.$$

In order to prove that $D \in {}^{\perp}\mathcal{C}$, we take an arbitrary extension

$$0 \longrightarrow C' \to X \xrightarrow{h} D \longrightarrow 0.$$

We will prove that h splits. First, consider the pullback of g and h:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C' & = & C' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & X & \longrightarrow & 0 \\
 & & \| & & \gamma\downarrow & & h\downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{f} & C & \xrightarrow{g} & D & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $C, C' \in \mathcal{C}$, also $P \in \mathcal{C}$ by assumption. Since f is a $\mathcal{C}$-envelope of M, there is a homomorphism $\delta : C \to P$ with $\alpha = \delta f$. Then $f = \gamma\alpha = \gamma\delta f$, so $\gamma\delta$ is an automorphism of C.

Define $i : D \to X$ by $i(g(c)) = \beta\delta(\gamma\delta)^{-1}(c)$. This is correct, since $\delta(\gamma\delta)^{-1} f(m) = \delta f(m) = \alpha(m)$. Moreover, $hig = h\beta\delta(\gamma\delta)^{-1} = g\gamma\delta(\gamma\delta)^{-1} = g$, so $hi = \mathrm{id}_D$ and h splits.

2. Dual to 1. ∎

Another reason for investigating special preenvelopes and precovers consists in their close relation to cotorsion theories:

Definition 1.10 Let $\mathcal{A}, \mathcal{B} \subseteq$ Mod-R. The pair $(\mathcal{A}, \mathcal{B})$ is called a *cotorsion theory* if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$.

If $\mathcal{C}$ is any class of modules, then

$$\mathfrak{G}_{\mathcal{C}} = ({}^{\perp}\mathcal{C}, ({}^{\perp}\mathcal{C})^{\perp})$$

and

$$\mathfrak{C}_{\mathcal{C}} = ({}^{\perp}(\mathcal{C}^{\perp}), \mathcal{C}^{\perp})$$

are easily seen to be cotorsion theories, called the cotorsion theory *generated* and *cogenerated*, respectively, by the class $\mathcal{C}$.

If $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion theory then the class $\mathcal{K}_{\mathfrak{C}} = \mathcal{A} \cap \mathcal{B}$ is called the *kernel* of $\mathfrak{C}$. Note that each element K of the kernel is a *splitter* in the sense of [30], i.e., $\mathrm{Ext}^1_R(K, K) = 0$.

For any ring R, the cotorsion theories of right R-modules are partially ordered by inclusion of their first components. In fact, they form a complete "lattice" L_{Ext}. In general, the support of L_{Ext} is a proper class [31] and L is not modular [43].

The largest element of L is $\mathfrak{G}_{\text{Mod-}R} = (\text{Mod-}R, \mathcal{I}_0)$, the least $\mathfrak{C}_{\text{Mod-}R} = (\mathcal{P}_0, \text{Mod-}R)$ - these are called the *trivial cotorsion theories*.

Cotorsion theories are analogs of the classical torsion theories where Hom is replaced by Ext. Similarly, one can define F-torsion theories for any additive bifunctor F on Mod-R.

The case when F is the Tor bifunctor is of particular interest for us. For a class of (right resp. left) R-modules, $\mathcal{C}$, we put

$$\mathcal{C}^{\intercal} = \mathrm{Ker}\,\mathrm{Tor}^R_1(\mathcal{C}, -) = \{N \in R\text{-Mod} \mid \mathrm{Tor}^R_1(C, N) = 0 \text{ for all } C \in \mathcal{C}\},$$

resp.

$${}^{\intercal}\mathcal{C} = \mathrm{Ker}\,\mathrm{Tor}^R_1(-, \mathcal{C}) = \{N \in \text{Mod-}R \mid \mathrm{Tor}^R_1(N, C) = 0 \text{ for all } C \in \mathcal{C}\}.$$

$(\mathcal{A},\mathcal{B})$ is called a *Tor-torsion theory* if $\mathcal{A} = {}^{\intercal}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\intercal}$.

Also Tor-torsion theories form a complete "lattice", L_{Tor}, with the least element $(\mathcal{FL}, \text{Mod-}R)$ and the largest $(\text{Mod-}R, \mathcal{FL})$.

Lemma 1.11 *Let R be a ring and $(\mathcal{A},\mathcal{B})$ be a Tor-torsion theory. Then $\mathfrak{C} = (\mathcal{A},\mathcal{A}^{\perp})$ is a cotorsion theory. Moreover, $\mathfrak{C} = \mathfrak{G}_{\mathcal{C}}$ where $\mathcal{C} = \{B^c \mid B \in \mathcal{B}\} \subseteq \mathcal{PI}$.*

Proof. The statement follows from the canonical isomorphism

$$\text{Ext}^1_R(A, \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}}(\text{Tor}^R_1(A,B), \mathbb{Q}/\mathbb{Z})$$

which holds for any $A \in \text{Mod-}R$ and $B \in R\text{-Mod}$, [12, VI.5.1]. ∎

Lemma 1.11 implies that there is a canonical order preserving embedding of L_{Tor} into L_{Ext}. In fact, the embedding is a lower "semilattice" one [43].

Definition 1.12 Consider the case of Lemma 1.11 when $\mathcal{A} = \mathcal{FL}$ and $\mathcal{B} = \text{Mod-}R$. Then $(\mathcal{FL}, \mathcal{EC})$ is a cotorsion theory, the so called *flat cotorsion theory*. Here, $\mathcal{EC} \overset{\text{def}}{=} \mathcal{FL}^{\perp}$ denotes the class of all *Enochs cotorsion modules*, [50]. By Lemma 1.11, any dual module, and hence any pure-injective module, is Enochs cotorsion. So $\mathcal{PI} \subseteq \mathcal{EC}$.

Another case of interest is when $\mathcal{A} = \mathcal{TF}$. Then $\mathcal{TF} = {}^{\intercal}\mathcal{S}$ where $\mathcal{S} = \{R/Rr \mid r \in R\}$. By Lemma 1.11, $(\mathcal{TF}, \mathcal{WC})$ is a cotorsion theory, the so called *torsion-free cotorsion theory*. Here, $\mathcal{WC} \overset{\text{def}}{=} \mathcal{TF}^{\perp}$ denotes the class of all *Warfield cotorsion modules*, [29, XII.3].

Clearly, $\mathcal{P}_0 \subseteq \mathcal{FL} \subseteq \mathcal{TF}$, so $\mathcal{I}_0 \subseteq \mathcal{WC} \subseteq \mathcal{EC}$ for any ring R.

Rather than looking at the structure of the lattice L_{Ext} we will be interested in aproximations induced by cotorsion theories. The basic fact is due to Salce [42]:

Lemma 1.13 *Let R be a ring and $\mathfrak{C} = (\mathcal{A},\mathcal{B})$ be a cotorsion theory of modules. Then the following are equivalent:*

1. *Each module has a special $\mathcal{A}$-precover;*

2. *Each module has a special $\mathcal{B}$-preenvelope.*

In this case, the cotorsion theory $\mathfrak{C}$ is called complete.

Proof.

1. implies 2.: Let $M \in \text{Mod-}R$. There is an exact sequence

$$0 \longrightarrow M \to I \xrightarrow{\pi} F \longrightarrow 0$$

where I is injective. By assumption, there is a special $\mathcal{A}$-precover, ρ, of F

$$0 \longrightarrow B \to A \xrightarrow{\rho} F \longrightarrow 0.$$

Consider the pullback of π and ρ:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & M & = & M & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & P & \longrightarrow & I & \longrightarrow & 0 \\ & & \| & & \gamma\downarrow & & \pi\downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & A & \xrightarrow{\rho} & F & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since $B, I \in \mathcal{B}$, also $P \in \mathcal{B}$. So the left-side vertical exact sequence is a special $\mathcal{B}$-preenvelope of M.

2. implies 1.: By a dual argument. ■

In Chapter 2, we will see that "almost all" cotorsion theories are complete, so they induce approximations of modules. In order to prove existence of minimal approximations, we will need the following version of a result due to Enochs and Xu [50]

Theorem 1.14 *Let R be a ring and M be a module. Let $\mathcal{C}$ be a class of modules closed under extensions and arbitrary direct limits. Assume that M has a special $\mathcal{C}^{\perp}$-preenvelope, ν, with* $\operatorname{Coker} \nu \in \mathcal{C}$. *Then M has a $\mathcal{C}^{\perp}$-envelope.*

Proof. 1. By an ad hoc notation, we will call an exact sequence $0 \longrightarrow M \to F \to C \longrightarrow 0$ with $C \in \mathcal{C}$ an *Ext-generator* provided that for each exact sequence $0 \longrightarrow M \to F' \to C' \longrightarrow 0$ with $C' \in \mathcal{C}$ there exist $f \in \operatorname{Hom}_R(F', F)$ and $g \in \operatorname{Hom}_R(C', C)$ such that the resulting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & f\downarrow & & g\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

is commutative. By assumption, there exists an Ext-generator with the middle term $F \in \mathcal{C}^{\perp}$. The proof is divided into three steps:

Lemma 1.15 *Assume* $0 \longrightarrow M \to F \to C \longrightarrow 0$ *is an Ext-generator. Then there exists an Ext-generator* $0 \longrightarrow M \to F' \to C' \longrightarrow 0$ *and a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \Big\| & & {\scriptstyle f}\Big\downarrow & & {\scriptstyle g}\Big\downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

such that $\mathrm{Ker}(f) = \mathrm{Ker}(f'f)$ *in any commutative diagram whose rows are Ext-generators:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\
 & & \Big\| & & {\scriptstyle f}\Big\downarrow & & {\scriptstyle g}\Big\downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \Big\| & & {\scriptstyle f'}\Big\downarrow & & {\scriptstyle g'}\Big\downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & F'' & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

Proof. Assume the assertion is not true. By induction, we will construct a directed system of Ext-generators indexed by ordinals as follows:

First, let the second row be the same as the first one, that is, put $F' = F_0 = F$, $C' = C_0 = C$, $f = \mathrm{id}_F$ and $g = \mathrm{id}_C$. Then there exist $F_1 = F''$, $C_1 = C''$, $f_{01} = f'$ and $g_{01} = g'$ such that the diagram above commutes, its rows are Ext-generators and $\mathrm{Ker}\, f_{10} \supsetneq \mathrm{Ker}\, f = 0$.

Assume the Ext-generator $0 \longrightarrow M \to F_\alpha \to C_\alpha \longrightarrow 0$ is defined together with $f_{\beta\alpha} \in \mathrm{Hom}_R(F_\beta, F_\alpha)$ and $g_{\beta\alpha} \in \mathrm{Hom}_R(C_\beta, C_\alpha)$, for all $\beta \leq \alpha$. Then there exist $F_{\alpha+1}$, $C_{\alpha+1} \in \mathcal{C}$, $f_{\alpha,\alpha+1}$ and $g_{\alpha,\alpha+1}$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F_\alpha & \longrightarrow & C_\alpha & \longrightarrow & 0 \\
 & & \Big\| & & {\scriptstyle f_{\alpha,\alpha+1}}\Big\downarrow & & {\scriptstyle g_{\alpha,\alpha+1}}\Big\downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & F_{\alpha+1} & \longrightarrow & C_{\alpha+1} & \longrightarrow & 0
\end{array}$$

commutes, its rows are Ext-generators and $\mathrm{Ker}\, f_{0,\alpha+1} \supsetneq \mathrm{Ker}\, f_{0\alpha}$, where $f_{\beta,\alpha+1} = f_{\alpha,\alpha+1} f_{\beta\alpha}$ and $g_{\beta,\alpha+1} = g_{\alpha,\alpha+1} g_{\beta\alpha}$ for all $\beta \leq \alpha$.

If α is a limit ordinal, put $F_\alpha = \varinjlim_{\beta<\alpha} F_\beta$ and $C_\alpha = \varinjlim_{\beta<\alpha} C_\beta$. Consider the direct limit $0 \longrightarrow M \to F_\alpha \to C_\alpha \longrightarrow 0$ of the Ext-generators $0 \longrightarrow M \to F_\beta \to C_\beta \longrightarrow 0$, $(\beta < \alpha)$. Since $\mathcal{C}$ is closed under direct limits, we have $C_\alpha \in \mathcal{C}$. Since $0 \longrightarrow M \to F_\beta \to C_\beta \longrightarrow 0$ is an Ext-generator for (some) $\beta < \alpha$, also $0 \longrightarrow M \to F_\alpha \to C_\alpha \longrightarrow 0$ is an Ext-generator.

Put $f_{\beta\alpha} = \varinjlim_{\beta\leq\beta'<\alpha} f_{\beta\beta'}$ and $g_{\beta\alpha} = \varinjlim_{\beta\leq\beta'<\alpha} g_{\beta\beta'}$ for all $\beta < \alpha$. Then $\mathrm{Ker}(f_{0\alpha}) \supseteq \mathrm{Ker}(f_{0\beta})$, and hence $\mathrm{Ker}(f_{0\alpha}) \supsetneq \mathrm{Ker}(f_{0\beta})$, for all $\beta < \alpha$.

By induction, we obtain for each ordinal α a strictly increasing chain, $(\mathrm{Ker}\, f_{0\beta} \mid \beta < \alpha)$, consisting of submodules of F, a contradiction. ∎

Lemma 1.16 *Assume* $0 \longrightarrow M \to F \to C \longrightarrow 0$ *is an Ext-generator. Then there exists an Ext-generator* $0 \longrightarrow M \to F' \to C' \longrightarrow 0$ *and a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

such that $\mathrm{Ker}(f') = 0$ *in any commutative diagram whose rows are Ext-generators:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f'}\downarrow & & {\scriptstyle g'}\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F'' & \longrightarrow & C'' & \longrightarrow & 0 \end{array}$$

Proof. By induction on $n < \omega$, we infer from Lemma 1.15 that there is a countable directed system, $\mathbb{D}$, of Ext-generators $0 \longrightarrow M \to F_n \to C_n \longrightarrow 0$ with homomorphisms $f_{n,n+1} \in \mathrm{Hom}_R(F_n, F_{n+1})$, $g_{n,n+1} \in \mathrm{Hom}_R(C_n, C_{n+1})$, such that the 0-th term of $\mathbb{D}$ is the given Ext-generator $0 \longrightarrow M \to F \to C \longrightarrow 0$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F_n & \longrightarrow & C_n & \longrightarrow & 0 \\ & & \| & & {\scriptstyle f_{n,n+1}}\downarrow & & {\scriptstyle g_{n,n+1}}\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & F_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0 \end{array}$$

is commutative, and for each commutative diagram whose rows are Ext-generators

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & F_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0 \\ & & \| & & {\scriptstyle \bar{f}}\downarrow & & {\scriptstyle \bar{g}}\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \bar{F} & \longrightarrow & \bar{C} & \longrightarrow & 0 \end{array}$$

we have $\mathrm{Ker}(f_{n,n+1}) = \mathrm{Ker}(\bar{f} f_{n,n+1})$.

Consider the direct limit $0 \longrightarrow M \to F' \to C' \longrightarrow 0$ of $\mathbb{D}$, so $F' = \varinjlim_{n<\omega} F_n$ and $C' = \varinjlim_{n<\omega} C_n$. Since $\mathcal{C}$ is closed under direct limits, we have $C' \in \mathcal{C}$, and $0 \longrightarrow M \to F' \to C' \longrightarrow 0$ is an Ext-generator. It is easy to check that this generator has the required injectivity property. ∎

Lemma 1.17 *Let* $0 \longrightarrow M \xrightarrow{\nu} F' \xrightarrow{\pi} C' \longrightarrow 0$ *be the Ext-generator constructed in Lemma 1.16. Then* $\nu : M \to F'$ *is a* $\mathcal{C}^{\perp}$*-envelope of* M*.*

Proof. First, we prove that that in each commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle f'}\downarrow & & {\scriptstyle g'}\downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & F' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

f' is an automorphism.

Assume this is not true. By induction, we construct a directed system of Ext-generators, $0 \longrightarrow M \to F_\alpha \to C_\alpha \longrightarrow 0$, indexed by ordinals, with injective, but not surjective, homomorphisms $f_{\beta\alpha} \in \mathrm{Hom}_R(F_\beta, F_\alpha)$ $(\beta < \alpha)$. In view of Lemma 1.16, we take

$$0 \longrightarrow M \to F_\alpha \to C_\alpha \longrightarrow 0 = 0 \longrightarrow M \xrightarrow{\nu} F' \xrightarrow{\pi} C' \longrightarrow 0$$

in case $\alpha = 0$ or α non-limit, and $F_\alpha = \varinjlim F_\beta$ and $C_\alpha = \varinjlim C_\beta$ if α is a limit ordinal. Then for each non-limit ordinal α, $(\mathrm{Im}\, g_{\beta\alpha} \mid \beta \text{ non-limit}, \beta < \alpha)$ is a strictly increasing sequence of submodules of F', a contradiction.

In remains to prove that $F' \in \mathcal{C}^\perp$. Consider an exact sequence $0 \longrightarrow F' \xrightarrow{\mu} X \to C \longrightarrow 0$ where $C \in \mathcal{C}$. We will prove that this sequence splits.

Consider the pushout of π and μ:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{\nu} & F' & \xrightarrow{\pi} & C' & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle \mu}\downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & P & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\mathcal{C}$ is closed under extensions, we have $P \in \mathcal{C}$. Since $0 \longrightarrow M \xrightarrow{\nu} F' \xrightarrow{\pi} C' \longrightarrow 0$ is an Ext-generator, we have also a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & P & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle \mu'}\downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{\nu} & F' & \xrightarrow{\pi} & C' & \longrightarrow & 0
\end{array}$$

By the first part of the proof, $\mu'\mu$ is an automorphism of F'. It follows that $0 \longrightarrow F' \xrightarrow{\mu} X \to C \longrightarrow 0$ splits. ■

■

Theorem 1.18 *Let R be a ring, M be a module, and $\mathcal{C}$ be a class of modules closed under arbitrary direct limits. Assume that M has a $\mathcal{C}$-precover. Then M has a $\mathcal{C}$-cover.*

Proof. The proof is by a construction of precovers with additional injectivity properties, in three steps analogous to Lemmas 1.15 - 1.17, cf. [50, §2.2] or [26]. ∎

Theorem 1.18 is not the strongest result available: for example, El Bashir has recently extended it to Grothendieck categories. Moreover, he proved that $\mathcal{C}$ is a cover class whenever $\mathcal{C}$ is a class of objects in a Grothendieck category $\mathcal{G}$ such that $\mathcal{C}$ is closed under arbitrary coproducts and directed colimits and there is a set of objects $\mathcal{S} \subseteq \mathcal{C}$ such that each object of $\mathcal{C}$ is a directed colimit of objects from $\mathcal{S}$, [24].

Corollary 1.19 *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a complete cotorsion theory such that $\mathcal{A}$ is closed under arbitrary direct limits. Then $\mathcal{A}$ is a cover class and $\mathcal{B}$ is an envelope class.*

Proof. By Theorems 1.14 and 1.18. ∎

2 Complete cotorsion theories

In this chapter, we will prove that complete cotorsion theories are abundant: any cotorsion theory cogenerated by a set of modules is complete, and so is any cotorsion theory generated by a class of pure-injective modules. In Chapter 3, we will apply these results to prove existence of various sorts of envelopes and covers of modules over arbitrary rings.

We start with a homological lemma. Let κ be an infinite cardinal. A chain of modules, $(M_\alpha \mid \alpha < \kappa)$, is called *continuous* provided that $M_\alpha \subseteq M_{\alpha+1}$ for all $\alpha < \kappa$ and $M_\alpha = \cup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha < \kappa$.

Lemma 2.1 *Let N be a module. Let $(M_\alpha \mid \alpha < \kappa)$ be a continuous chain of modules. Put $M = \cup_{\alpha<\kappa} M_\alpha$.*

Assume that $\operatorname{Ext}^1_R(M_0, N) = 0$ and $\operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) = 0$ for all $\alpha < \kappa$. Then $\operatorname{Ext}^1_R(M, N) = 0$.

Proof. Put $M_\kappa = M$. By induction on $\alpha \leq \kappa$, we will prove that $\operatorname{Ext}^1_R(M_\alpha, N) = 0$. By assumption, this is true for $\alpha = 0$.

The exact sequence $0 = \operatorname{Ext}^1_R(M_{\alpha+1}/M_\alpha, N) \to \operatorname{Ext}^1_R(M_{\alpha+1}, N) \to \operatorname{Ext}^1_R(M_\alpha, N) = 0$ proves the induction step.

Assume $\alpha < \kappa$ is a limit ordinal. Let $0 \longrightarrow N \to I \xrightarrow{\pi} I/N \longrightarrow 0$ be an exact sequence with I an injective module. In order to prove that $\mathrm{Ext}^1_R(M_\alpha, N) = 0$, we show that the abelian group homomorphism $\mathrm{Hom}_R(M_\alpha, \pi) : \mathrm{Hom}_R(M_\alpha, I) \to \mathrm{Hom}_R(M_\alpha, I/N)$ is surjective.

Let $\varphi \in \mathrm{Hom}_R(M_\alpha, I/N)$. By induction, we define homomorphisms $\psi_\beta \in \mathrm{Hom}_R(M_\beta, I/N)$, $\beta < \alpha$, so that $\varphi \restriction M_\beta = \pi\psi_\beta$ and $\psi_\beta \restriction M_\gamma = \psi_\gamma$ for all $\gamma < \beta < \alpha$.

First, define $M_{-1} = 0$ and $\psi_{-1} = 0$. If ψ_β is already defined, the injectivity of I yields the existence of $\eta \in \mathrm{Hom}_R(M_{\beta+1}, I)$ such that $\eta \restriction M_\beta = \psi_\beta$. Put $\delta = \varphi \restriction M_{\beta+1} - \pi\eta \in \mathrm{Hom}_R(M_{\beta+1}, I/N)$. Then $\delta \restriction M_\beta = 0$. Since $\mathrm{Ext}^1_R(M_{\beta+1}/M_\beta, N) = 0$, there is $\epsilon \in \mathrm{Hom}_R(M_{\beta+1}, I)$ such that $\epsilon \restriction M_\beta = 0$ and $\pi\epsilon = \delta$. Put $\psi_{\beta+1} = \eta + \epsilon$. Then $\psi_{\beta+1} \restriction M_\beta = \psi_\beta$ and $\pi\psi_{\beta+1} = \pi\eta + \delta = \varphi \restriction M_{\beta+1}$. For a limit ordinal $\beta < \alpha$, put $\psi_\beta = \cup_{\gamma<\beta}\psi_\gamma$

Finally, put $\psi_\alpha = \cup_{\beta<\alpha}\psi_\beta$. By the construction, $\pi\psi_\alpha = \varphi$.

The claim is just the case of $\alpha = \kappa$. ∎

The next theorem is crucial. It was originally proved in [22] which in turn generalized a particular construction for torsion-free abelian groups [30]. The proof given here is a more categorical modification of the original proof, cf. [1]:

Theorem 2.2 *Let $\mathcal{S}$ be a set of modules.*

1. *Let M be a module. Then there is a short exact sequence $0 \to M \hookrightarrow P \to N \to 0$ where $P \in \mathcal{S}^\perp$ and P is the union of a continuous chain of submodules, $(P_\alpha \mid \alpha < \lambda)$, such that $P_0 = M$ and $P_{\alpha+1}/P_\alpha$ is isomorphic to a direct sum of copies of elements of $\mathcal{S}$ for each $\alpha < \lambda$.*

   In particular, $M \hookrightarrow P$ is a special $\mathcal{S}^\perp$-preenvelope of M.

2. *The cotorsion theory $\mathfrak{C}_{\mathcal{S}} = ({}^\perp(\mathcal{S}^\perp), \mathcal{S}^\perp)$ is complete.*

.

Proof. 1. Put $X = \oplus_{S\in\mathcal{S}}S$. Then $X^\perp = \mathcal{S}^\perp$. So w.l.o.g., we assume that $\mathcal{S}$ consists of a single module S.

Let $0 \longrightarrow K \xrightarrow{\mu} F \to S \longrightarrow 0$ be a short exact sequence with F a free module. Let λ be an infinite regular cardinal such that K is $< \lambda$-generated.

By induction, we define the chain $(P_\alpha \mid \alpha < \lambda)$ as follows:

First $P_0 = M$. For $\alpha < \lambda$, define μ_α as the direct sum of $\mathrm{Hom}_R(K, P_\alpha)$-many copies of μ, so

$$\mu_\alpha = \bigoplus \mu \in \mathrm{Hom}_R(K^{(\mathrm{Hom}_R(K,P_\alpha))}, F^{(\mathrm{Hom}_R(K,P_\alpha))}).$$

Then μ_α is a monomorphism and $\mathrm{Coker}\,\mu_\alpha$ is isomorphic to a direct sum of copies of S. Let $\varphi_\alpha \in \mathrm{Hom}_R(K^{(\mathrm{Hom}_R(K,P_\alpha))}, P_\alpha)$ be the canonical morphism. Note that for each $\eta \in \mathrm{Hom}_R(K, P_\alpha)$, there exist canonical embeddings $\nu_\eta \in \mathrm{Hom}_R(K, K^{(\mathrm{Hom}_R(K,P_\alpha))})$ and $\nu'_\eta \in \mathrm{Hom}_R(F, F^{(\mathrm{Hom}_R(K,P_\alpha))})$ such that $\eta = \varphi_\alpha\nu_\eta$ and $\nu'_\eta\mu = \mu_\alpha\nu_\eta$.

Now, $P_{\alpha+1}$ is defined via the pushout of μ_α and φ_α:

$$\begin{array}{ccc} K^{(\mathrm{Hom}_R(K,M_\alpha))} & \xrightarrow{\mu_\alpha} & P^{(\mathrm{Hom}_R(K,M_\alpha))} \\ \varphi_\alpha \downarrow & & \psi_\alpha \downarrow \\ P_\alpha & \xrightarrow{\subseteq} & P_{\alpha+1} \end{array}$$

If $\alpha \leq \lambda$ is a limit ordinal, we put $P_\alpha = \cup_{\beta<\alpha} P_\beta$, so the chain is continuous. Put $P = \cup_{\alpha<\lambda} P_\alpha$.

We will prove that $\nu : M \hookrightarrow P$ is a special $S^\perp$-preenvelope of M.

First, we prove that $P \in S^\perp$. Since F is projective, we are left to show that any $\varphi \in \mathrm{Hom}_R(K, P)$ factors through μ:

Since K is $< \lambda$-generated, there are an index $\alpha < \lambda$ and $\eta \in \mathrm{Hom}_R(K, P_\alpha)$ such that $\varphi(k) = \eta(k)$ for all $k \in K$. The pushout square gives $\psi_\alpha \mu_\alpha = \sigma_\alpha \varphi_\alpha$, where σ_α denotes the inclusion of P_α into $P_{\alpha+1}$. Altogether, we have $\psi_\alpha \nu'_\eta \mu = \psi_\alpha \mu_\alpha \nu_\eta = \sigma_\alpha \varphi_\alpha \nu_\eta = \sigma_\alpha \eta$. It follows that $\varphi = \psi' \mu$ where $\psi' \in \mathrm{Hom}_R(F, P)$ is defined by $\psi'(f) = \psi_\alpha \nu'_\eta(f)$ for all $f \in F$. This proves that $P \in S^\perp$.

It remains to prove that $N = P/M \in {}^\perp(\mathcal{S}^\perp)$. By the construction, N is the union of the continuous chain $(N_\alpha \mid \alpha < \lambda)$ where $N_\alpha = P_\alpha/M$.

Since $P_{\alpha+1}/P_\alpha$ is isomorphic to a direct sum of copies of S by the pushout construction, so is $N_{\alpha+1}/N_\alpha \cong P_{\alpha+1}/P_\alpha$. Since $S \in {}^\perp(\mathcal{S}^\perp)$, Lemma 2.1 shows that $N \in {}^\perp(\mathcal{S}^\perp)$.

2. Follows by part 1 (cf. Lemma 1.13). ∎

Again, Theorem 2.2 is not the most general result available: for example, $\mathcal{C}$ is a preenvelope class whenever $\mathcal{C}$ is the class of all objects injective w.r.t. a set of monomorphisms in a Grothendieck category, [46].

From Theorem 2.2, we easily get a characterization of the (complete) cotorsion theories cogenerated by sets of modules:

Corollary 2.3 *Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion theory. Then the following are equivalent*

1. *$\mathfrak{C}$ is cogenerated by a set of modules.*
2. *There is a module M such that $\mathcal{A}$ consists of all direct summands, A, of modules of the form $Z = Z_\lambda$ where λ is a cardinal, $(Z_\alpha \mid \alpha \leq \lambda)$ is a continuous chain, Z_0 is a free module and for each $\alpha < \lambda$, $Z_{\alpha+1}/Z_\alpha \cong M$. Moreover, Z can be taken so that there exists $P \in \mathcal{K}_\mathfrak{C}$ such that $A \oplus P \cong Z$.*

Proof. 1. implies 2.: We have $\mathcal{B} = M^\perp$ for a module M. Take $A \in \mathcal{A}$ and let $0 \longrightarrow N \xrightarrow{\mu} F \to A \longrightarrow 0$ be a short exact sequence with F free. By Theorem 2.2.1, there is a special $\mathcal{B}$-preenvelope, $0 \longrightarrow N \xrightarrow{\nu} P \to P/N \longrightarrow 0$

of N, such that P/N is a union of a continuous chain, $(P_\alpha/N \mid \alpha < \lambda)$, with successive factors isomorhic to M. Consider the pushout of μ and ν:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & N & \xrightarrow{\mu} & F & \longrightarrow & A & \longrightarrow & 0 \\ & & {\scriptstyle\nu}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & P & \longrightarrow & Z & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & {\scriptstyle\pi}\downarrow & & & & \\ & & P/N & = & P/N & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Then $Z = \cup_{\alpha<\lambda} Z_\alpha$ where Z_α are the pre-images of P_α/N in π. So $Z_0 = F$ and the succesive factors $Z_{\alpha+1}/Z_\alpha$ are isomorphic to M. The second row splits since $P \in \mathcal{B}$ and $A \in \mathcal{A}$, so $A \oplus P \cong Z$. Finally, since $F, P/N \in \mathcal{A}$, we have $Z \in \mathcal{A}$, so $P \in \mathcal{K}_{\mathfrak{C}}$.

2. implies 1. By Lemma 2.1, since both $R \in \mathcal{A}$ and $M \in \mathcal{A}$. ∎

Though we will see that many cotorsion theories satisfy the equivalent conditions of Corollary 2.3, this is not always the case:

Example 2.4 Let $R = \mathbb{Z}$ and $\mathfrak{W} = ({}^\perp\mathbb{Z}, ({}^\perp\mathbb{Z})^\perp)$ (note that ${}^\perp\mathbb{Z}$ is the class of all Whitehead groups). Using the model of ZFC with Shelah's uniformization principle UP [20], [44], Eklof and Shelah have recently proved that it is consistent with ZFC + GCH that $\mathfrak{W}$ is not cogenerated by a set, see [21].

This consistency result is not provable in ZFC: Eklof and the author proved that assuming Gödel's axiom of constructibility (V = L), $\mathfrak{C}$ is complete whenever $\mathfrak{C}$ is a cotorsion theory generated by a set of modules over a right hereditary ring, [23].

Important examples of complete cotorsion theories are provided by cotorsion theories generated by classes of pure-injective modules. Before proving this, we require a definition, and a lemma on purity, following [23, §3].

Definition 2.5 For any module A and any cardinal κ, a *κ-refinement of A* (of length σ) is an increasing sequence $(A_\alpha \mid \alpha \leq \sigma)$ of pure submodules of A such that $A_0 = 0$, $A_\sigma = A$, $A_\alpha = \cup_{\beta<\alpha} A_\beta$ for all limit ordinals $\alpha \leq \sigma$, and $\mathrm{card}(A_{\alpha+1}/A_\alpha) \leq \kappa$ for all $\alpha + 1 \leq \sigma$.

Lemma 2.6 *Let $\kappa \geq card(R) + \aleph_0$.*

1. *Let M be a module and X be a subset of M with card$(X) \leq \kappa$. Then there is a pure submodule $N \subseteq_* M$ such that $X \subseteq N$ and card$(N) \leq \kappa$.*

2. *Assume $C \subseteq_* A$ and $B/C \subseteq_* A/C$. Then $B \subseteq_* A$.*

3. *Assume $A_0 \subseteq \cdots \subseteq A_\alpha \subseteq A_{\alpha+1} \subseteq \ldots$ is a chain of pure submodules of M. Then $\cup_\alpha A_\alpha$ is a pure submodule of M.*

Proof. Well-known (see [36, Theorem 6.4]). ▪

Lemma 2.7 *Let $\kappa = card(R) + \aleph_0$. Let $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ be a cotorsion theory generated by a class $\mathcal{C} \subseteq \mathcal{PI}$. Then the following are equivalent*

1. *$A \in \mathcal{A}$.*

2. *There is a cardinal λ such that A has a κ-refinement $(A_\alpha \mid \alpha \leq \lambda)$ with $A_{\alpha+1}/A_\alpha \in \mathcal{A}$ for all $\alpha < \lambda$.*

Proof. 1. implies 2.: If card$(A) \leq \kappa$, we let $\lambda = 1$, $A_0 = 0$, and $A_1 = A$. So we can assume that card$(A) > \kappa$. Let $\lambda =$ card(A). Then $A \cong F/K$ where $F = R^{(\lambda)}$ is a free module. We enumerate the elements of F in a λ-sequence: $F = \{x_\alpha \mid \alpha < \lambda\}$. By induction on α, we will define a sequence $(A_\alpha \mid \alpha \leq \lambda)$ so that for all $\alpha \leq \lambda$, A_α is pure in A and belongs to ${}^\perp\mathcal{C}$. Since each $C \in \mathcal{C}$ is pure-injective, it will follow from the long exact sequence induced by

$$0 \to A_\alpha \to A_{\alpha+1} \to A_{\alpha+1}/A_\alpha \to 0$$

that $A_{\alpha+1}/A_\alpha \in \mathcal{A}$ for all $\alpha < \lambda$.

A_α will be constructed so that it equals $(R^{(I_\alpha)} + K)/K$ for some $I_\alpha \subseteq \lambda$ such that $R^{(I_\alpha)} \cap K$ is pure in K. Let $A_0 = 0$. Assume A_β has been defined for all $\beta < \sigma$. Suppose first that $\sigma = \alpha + 1$. By induction on $n < \omega$ we will define an increasing chain $F_0 \subseteq F_1 \subseteq \ldots$ and then put $A_{\alpha+1} = \cup_{n<\omega}(F_n + K)/K$. We require that card$(F_{n+1}/F_n) \leq \kappa$ for all $n < \omega$, and furthermore: for n odd, $(F_n + K)/K$ is pure in F/K; for n even, $F_n = R^{(J_n)}$ for some $J_n \supseteq J_{n-2} \supseteq \cdots \supseteq J_0$ and $F_n \supseteq K'_n$ where $F_{n-1} \cap K \subseteq K'_n \subseteq_* K$.

First, put $F_0 = R^{(I_\alpha)}$ and let $J_0 = I_\alpha$ and $K'_0 = R^{(I_\alpha)} \cap K$. Assume F_{n-1} has been constructed and n is odd. By part 1. of Lemma 2.6 there is a pure submodule $(F_n + K)/(F_{n-2} + K) \subseteq_* F/(F_{n-2} + K)$ of cardinality $\leq \kappa$ containing $(x_\alpha R + F_{n-1} + K)/(F_{n-2} + K)$. Moreover, we can choose F_n so that card$(F_n/F_{n-1}) \leq \kappa$. By part 2. of Lemma 2.6, $(F_n + K)/K$ is pure in F/K.

Assume $n > 0$ is even. We first define K'_n: by part 1. of Lemma 2.6, we find a pure submodule $K'_n/K'_{n-2} \subseteq_* K/K'_{n-2}$ of cardinality $\leq \kappa$ containing $(F_{n-1} \cap K)/K'_{n-2}$. This is possible since $K'_{n-2} \supseteq F_{n-3} \cap K$ and $(F_{n-1} \cap$

$K)/(F_{n-3} \cap K)$ embeds in F_{n-1}/F_{n-3}, so it has cardinality $\leq \kappa$. By part 2. of Lemma 2.6, we have $K'_n \subseteq_* K$.

We can choose $J_n \subseteq \lambda$ such that $\mathrm{card}(J_n - J_{n-2}) \leq \kappa$ and $F_{n-1} + K'_n \subseteq R^{(J_n)} = F_n$. This is possible since $\mathrm{card}((F_{n-1} + K'_n)/F_{n-2}) \leq \kappa$; indeed, we have the exact sequence

$$0 \to F_{n-1}/F_{n-2} \to (F_{n-1} + K'_n)/F_{n-2} \to (F_{n-1} + K'_n)/F_{n-1} \to 0$$

and $(F_{n-1} + K'_n)/F_{n-1} \cong K'_n/(F_{n-1} \cap K)$ has cardinality $\leq \kappa$ because it is a homomorphic image of K'_n/K'_{n-2}.

Now, define $A_{\alpha+1} = \cup_{n<\omega}(F_n + K)/K$ and $I_{\alpha+1} = \cup_{n<\omega} J_{2n}$. By part 3. of Lemma 2.6, $A_{\alpha+1} \subseteq_* A$. Clearly, $\mathrm{card}(A_{\alpha+1}/A_\alpha) \leq \kappa$.

We have $A_{\alpha+1} \cong F'/K'$, where $F' = \cup_{n<\omega} F_{2n}$ and $K' = F' \cap K$. Also, $F' = R^{(I_{\alpha+1})}$ is free, and $K' = \cup_{n<\omega} K'_{2n}$ is pure in K by construction and part 3. of Lemma 2.6.

Let $C \in \mathcal{C}$. In order to prove that $\mathrm{Ext}(A_{\alpha+1}, C) = 0$, we have to extend any $f \in \mathrm{Hom}(K', C)$ to an element of $\mathrm{Hom}(F', C)$. First, f extends to K, since $K' \subseteq_* K$ and C is pure-injective. By the assumption 1., we can extend further to F, and then restrict to F'.

Finally, if $\sigma \leq \lambda$ is a limit ordinal, let $A_\sigma = \cup_{\beta<\sigma} A_\beta$; that A_σ has the desired properties follows from Lemma 2.1 and part 3. of Lemma 2.6.

2. implies 1.: By Lemma 2.1. ∎

Theorem 2.8 *Let $\mathfrak{G}_{\mathcal{C}} = (\mathcal{A}, \mathcal{B})$ be a cotorsion theory generated by a class $\mathcal{C} \subseteq \mathcal{PI}$. Then $\mathfrak{G}_{\mathcal{C}}$ is complete. Moreover, $\mathcal{A}$ is a cover class and $\mathcal{B}$ is an envelope class.*

Proof. Let $\kappa = \mathrm{card}(R) + \aleph_0$. Denote by H the direct sum of a representative set of the class $\{A \in \text{Mod-}R \mid \mathrm{card}(A) \leq \kappa \,\&\, \mathrm{Ext}(A, \mathcal{C}) = 0\}$. Clearly, $\mathcal{B} \subseteq H^\perp$. Conversely, take $D \in H^\perp$. Let $A \in \mathcal{A}$; by Lemma 2.7, A has a κ-refinement $(A_\alpha \mid \alpha \leq \lambda)$. By the choice of H, $\mathrm{Ext}(A_{\alpha+1}/A_\alpha, D) = 0$ for all $\alpha < \lambda$ and hence, by Lemma 2.1, $\mathrm{Ext}(A, D) = 0$. So $D \in \mathcal{B}$. This proves that $\mathcal{B} = H^\perp$. By Theorem 2.2.2, $\mathfrak{G}_{\mathcal{C}}$ is a complete cotorsion theory.

By Corollary 1.19, it remains to show that the class $\mathcal{A}$ is closed under arbitrary direct limits. Assume $P \in \mathcal{PI}$. Then ${}^\perp P$ is closed under homomorphic images of pure epimorphisms. But the canonical map of a direct sum on to a direct limit is well-known to be a pure epimorphism (cf. [49, 33.9]). So ${}^\perp P$ is closed under arbitrary direct limits, and so is $\mathcal{A} = {}^\perp \mathcal{C}$. ∎

There is an analogue of Lemma 2.7 for the bifunctor Tor:

Lemma 2.9 *Let $\mathcal{C}$ be any class of left R-modules. Let κ = card$(R) + \aleph_0$. The following conditions are equivalent for any module A:*

1. $A \in {}^{\intercal}\mathcal{C}$,

2. *there is a cardinal λ such that A has a κ-refinement $(A_\alpha \mid \alpha \leq \lambda)$ such that $A_{\alpha+1}/A_\alpha \in {}^{\intercal}\mathcal{C}$ for all $\alpha < \lambda$.*

Proof. Put $\mathcal{P} = \{C^c \mid C \in \mathcal{C}\}$. Then $\mathcal{P}$ is a class of pure-injective modules and ${}^{\perp}\mathcal{P} = {}^{\intercal}\mathcal{C}$ by Lemma 1.11. So the assertion follows from Lemma 2.7. ■

Theorem 2.10 *1. Let $\mathcal{C}$ be any class of left R-modules. Then every module has a ${}^{\intercal}\mathcal{C}$-cover.*

2. Let $\mathcal{D}$ be any class consisting of character modules (of left R-modules). Then every module has a ${}^{\perp}\mathcal{D}$-cover.

Proof. 1. As above, we have $\mathcal{A} = {}^{\intercal}\mathcal{C} = {}^{\perp}\mathcal{P}$ where $\mathcal{P}$ is a class of pure-injective modules. Then every module has an $\mathcal{A}$-cover by Theorem 2.8.

2. Since any character module is pure-injective, every module has a ${}^{\perp}\mathcal{D}$-cover by Theorem 2.8. ■

Example 2.11 1. Let k be a field and R be a k-algebra. Let $\mathcal{M}$ be a class of k-finite dimensional modules. Then every module has a ${}^{\perp}\mathcal{M}$-cover. Indeed, any k-finite dimensional module is dual (in the k-vector space duality), hence pure-injective, and Theorem 2.8 applies.

2. Assume that R is a right pure-semisimple ring. Let $\mathcal{C}$ be any class of modules. Then every module has a ${}^{\perp}\mathcal{C}$-cover. This is because every R-module is pure-injective (see [36, Theorem 8.4]), so Theorem 2.8 applies once again.

3 A proof of the FCC and further applications

Now, we are in a position to construct at once various approximations of modules over arbitrary rings. We start with a proof of the Flat Cover Conjecture, and with a generalization of the Enochs' construction [25] of torsion-free covers of modules over domains, cf. [9], [45]:

Theorem 3.1 *1. Every module has a flat cover and an Enochs cotorsion envelope.*

2. Every module has a torsion-free cover and a Warfield cotorsion envelope.

Proof. We have $\mathcal{FL} = {}^{\perp}\mathcal{PI}$, and $\mathcal{TF} = {}^{\perp}\mathcal{D}$ where $\mathcal{D} = \{N^c \mid N = R/Rr \,\&\, r \in R\}$ (cf. Lemma 1.11 and Definition 1.12). So Theorem 2.8 applies to the cotorsion theories $(\mathcal{FL}, \mathcal{EC})$ and $(\mathcal{TF}, \mathcal{WC})$, respectively. ■

Example 3.2 Let R be a domain. A module M is *Matlis cotorsion* provided that $\mathrm{Ext}^1_R(Q, M) = 0$, [39]. For example, if M is a reduced torsion-free module then $M \in \mathcal{MC}$ iff M is R-complete, [29, Proposition V.1.2]. Similarly, if M is *bounded* (i.e., there exists $0 \neq r \in R$ with $Mr = 0$) then $M \in \mathcal{MC}$, [29, XII.3.3].

Denote by $\mathcal{MC}$ the class of all Matlis cotorsion modules. Since Q is a flat module (namely, a localization of R at 0) we have $\mathcal{WC} \subseteq \mathcal{EC} \subseteq \mathcal{MC}$.

The coincidence of these classes characterizes the Prüfer and Dedekind domains: R is Prüfer iff $\mathcal{FL} = \mathcal{TF}$ iff $\mathcal{EC} = \mathcal{WC}$, [29, XII.3]. By [39], R is Dedekind iff $\mathcal{WC} = \mathcal{MC}$. In fact, for any domain $\mathcal{WC} = \mathcal{MC} \cap \mathcal{I}_1$, [29, XII.3].

By Theorem 2.2, $({}^{\perp}\mathcal{MC}, \mathcal{MC})$ is a complete cotorsion theory. Moreover, Mod-Q is a subclass of Mod-R closed under extensions and arbitrary direct limits, Mod-$Q^{\perp} = \mathcal{MC}$, and Q is a $\sum$-injective module. By Theorems 2.2.1 and 1.14, we infer that *each module has an $\mathcal{MC}$-envelope.*

In some cases, there is a more explicit description of the cotorsion envelopes: if M is reduced and torsion-free then the $\mathcal{MC}$-envelope is just the inclusion $M \hookrightarrow \hat{M}$ where $\hat{M}$ is the R-completion of M, [45], [29, V.2]. The $\mathcal{WC}$-envelope of any module M coincides with the inclusion $M \hookrightarrow \bar{M}$ where $\bar{M}$ denotes the Warfield cotorsion hull of M, cf. Definition 1.12, Lemma 1.17 and [29, XII.4].

Next, we prove existence of special divisible and FP-injective preenvelopes of modules:

Definition 3.3 A module is *FP-injective* provided that $\mathrm{Ext}^1_R(F, M) = 0$ for each finitely presented module F. Denote by $\mathcal{FI}$ the class of all FP-injective modules.

A module M is *cyclically covered* (*finitely covered*) provided that M is a direct summand in a module N such that N is a union of a continuous chain, $(N_\alpha \mid \alpha < \lambda)$, for a cardinal λ, $N_0 = 0$, and $N_{\alpha+1}/N_\alpha$ is cyclically presented (finitely presented) for all $\alpha < \lambda$. Denote by $\mathcal{CC}$ ($\mathcal{FC}$) the class of all cyclically covered (finitely covered) modules.

Clearly, $\mathcal{I}_0 \subseteq \mathcal{FI} \subseteq \mathcal{DI}$ and $\mathcal{CC} \subseteq \mathcal{FC}$ for any ring R. By Auslander lemma, $\mathcal{CC} \subseteq P_1$ in the case when R is a domain. Moreover, $\mathcal{FC} = \mathcal{P}_1$ for any Prüfer domain by [29, Corollary IV.4.7].

Theorem 3.4 *1. $(\mathcal{CC}, \mathcal{DI})$ is a complete cotorsion theory. In particular, every module has a special divisible preenvelope.*

2. $(\mathcal{FC},\mathcal{FI})$ is a complete cotorsion theory. In particular, every module has a special FP-injective preenvelope.

Proof. Put $M = \bigoplus_{r\in R} R/rR$ and let N be the direct sum of a representative set of all finitely presented modules. By Theorem 2.2 and Corollary 2.3, $(\mathcal{CC},\mathcal{DI})$ and $(\mathcal{FC},\mathcal{FI})$ are complete cotorsion theories cogenerated by M and N, respectively. ∎

The existence of special divisible preenvelopes in the particular case of modules over domains was essentially proved in [29, VI.3]. In Proposition 4.8 and Theorem 4.9, we will see that the statement of Theorem 3.4 is the best possible in the sense that there exist no divisible envelopes, and no FP-injective envelopes, in general.

Next, we consider approximations by modules of a finite homological dimension:

Theorem 3.5 *Let $n < \omega$. Then $({}^{\perp}\mathcal{I}_n,\mathcal{I}_n)$ is a complete cotorsion theory. In particular, every module has a special $\mathcal{I}_n$-preenvelope.*

Proof. Let M be a module. Let

$$0 \to N \to I_0 \xrightarrow{f_0} I_1 \xrightarrow{f_1} \cdots \to I_{n-1} \xrightarrow{f_{n-1}} I_n \xrightarrow{f_n} \dots$$

be an injective resolution of M. Put $S_n = \operatorname{Im} f_{n-1} = \operatorname{Ker} f_n$. Then $N \in \mathcal{I}_n$ iff S_n is injective. By Baer's criterion, the latter is equivalent to $\operatorname{Ext}^1_R(R/I, S_n) = 0$, and hence – by dimension shifting – to $\operatorname{Ext}^n_R(R/I, M) = 0$, for all right ideals I of R. Denote by C_I the n-th syzygy module (in a projective resolution) of the cyclic module R/I. Then $\operatorname{Ext}^n_R(R/I, M) = 0$ iff $\operatorname{Ext}^1_R(C_I, M) = 0$. So $\mathcal{I}_n = (\bigoplus_{I\subseteq R} C_I)^{\perp}$, and the assertion follows by Theorem 2.2.2. ∎

Theorem 3.5 was proved in the particular case when R is right noetherian in [1, Proposition 3.1]. Our general proof is based on the existence of a test module for injectivity, that is, on Baer's criterion.

The corresponding result for $(\mathcal{P}_n,\mathcal{P}_n^{\perp})$ can be proved dually in the case when there exists a test module for projectivity: this happens when R is right perfect or, under V = L, when R is right hereditary, cf. [44]. Nevertheless, if R is not right perfect then the Shelah's uniformization principle UP implies that there are no such test modules at all [44]. So one needs a different approach of [1]:

Lemma 3.6 *Let R be a ring and $n < \omega$. Let $\kappa = card(R)+\aleph_0$. Let $M \in \mathcal{P}_n$. Then there are a cardinal λ and a continuous chain, $(M_\alpha \mid \alpha < \lambda)$, consisting of submodules of M such that $M = \cup_{\alpha<\lambda} M_\alpha$, $M_\alpha, M_{\alpha+1}/M_\alpha \in \mathcal{P}_n$ and $card(M_{\alpha+1}/M_\alpha) \leq \kappa$ for all $\alpha < \lambda$.*

Proof. Let

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

be a projective resolution of M. By the Kaplansky Theorem [2], each projective module is a direct sum of countably generated modules, so $P_i = \oplus_{\alpha<\lambda_i} P_{i\alpha}$ where $P_{i\alpha}$ is countably generated for all $i \leq n$ and $\alpha < \lambda_i$.

Let $0 \neq x \in M$. Then there is a finite subset $F_0 \subseteq \lambda_0$ such that $x \in f_0(\oplus_{j\in F_0} P_{0j})$. Further, there is a countable subset $F_1 \subseteq \lambda_1$ such that $\mathrm{Ker}(f_0 \restriction \oplus_{j\in F_0} P_{0j}) \subseteq f_1(\oplus_{j\in F_1} P_{1j})$. Similarly, there is a countable subset $F_2 \subseteq \lambda_2$ such that $\mathrm{Ker}(f_1 \restriction \oplus_{j\in F_1} P_{1j}) \subseteq f_2(\oplus_{j\in F_2} P_{2j})$, etc. Finally, there is a countable subset $F_n \subseteq \lambda_n$ such that $\mathrm{Ker}(f_{n-1} \restriction \oplus_{j\in F_{n-1}} P_{n-1,j}) \subseteq f_n(\oplus_{j\in F_n} P_{nj})$. Now, there is a countable subset $F_{n-1} \subseteq F'_{n-1} \subseteq \lambda_{n-1}$ such that $f_n(\oplus_{j\in F_n} P_{nj}) \subseteq \oplus_{j\in F'_{n-1}} P_{n-1,j}$ etc. Finally, there is a countable subset $F_0 \subseteq F'_0 \subseteq \lambda_0$ such that $f_1(\oplus_{j\in F'_1} P_{1j}) \subseteq \oplus_{j\in F'_0} P_{0j}$. Continuing this back and forth procedure, we obtain for each $i \leq n$ a countable subset $C_i = F_i \cup F'_i \cup F''_i \cup \dots$ of λ_i such that the restricted sequence

$$0 \to \oplus_{j\in C_n} P_{nj} \xrightarrow{f_n} \oplus_{j\in C_{n-1}} P_{n-1,j} \to \dots$$
$$\dots \to \oplus_{j\in C_1} P_{1j} \xrightarrow{f_1} \oplus_{j\in C_0} P_{0j} \xrightarrow{f_0} N \to 0$$

is exact, and $x \in N \subseteq M$. By the construction, also the factor-sequence

$$0 \to \oplus_{j\notin C_n} P_{nj} \xrightarrow{\bar{f}_n} \oplus_{j\notin C_{n-1}} P_{n-1,j} \to \dots$$
$$\dots \to \oplus_{j\notin C_1} P_{1j} \xrightarrow{\bar{f}_1} \oplus_{j\notin C_0} P_{0j} \xrightarrow{\bar{f}_0} M/N \to 0$$

is exact. So $N, M/N \in \mathcal{P}_n$. Since $\mathrm{card}(N) \leq \kappa$, we put $M_0 = N$ and proceed similarly, constructing the required continuous chain by induction; in limit steps, we use the fact that $\mathcal{P}_n$ is closed under unions of chains by Auslander lemma. ∎

Theorem 3.7 *Let $n < \omega$. Then $(\mathcal{P}_n, \mathcal{P}_n^\perp)$ is a complete cotorsion theory. In particular, every module has a special $\mathcal{P}_n$-precover.*

Proof. By Lemmas 2.1 and 3.6, we have $M_n^\perp = \mathcal{P}_n^\perp$ where M_n denotes the direct sum of a representative set of those modules from $\mathcal{P}_n$ which have cardinality $\leq \kappa$. Now, the assertion follows by Theorem 2.2.2. ∎

Though $(\mathcal{P}_n, \mathcal{P}_n^\perp)$ is always complete, there may be no minimal approximations available: By a well-known result of Bass, $\mathcal{P}_0$ is a cover class iff R is right perfect [2], while trivially $\mathcal{P}_0^\perp$ is always an envelope class. For the case of $n = 1$, see Theorem 4.9.

Theorem 3.7 is no more true when we replace Mod-R by the category of all finite dimensional modules – counter-examples for each $n \geq 1$ were constructed in [35].

If R is right noetherian then there is a cardinal κ such that each injective module is a direct sum of $\leq \kappa$ generated modules [2], and $\mathcal{I}_n$ is closed under direct limits. So the analog of Lemma 3.6 holds true for $\mathcal{I}_n$ - the proof is just dual to the one given in 3.6. By Theorems 2.2.2, 1.14 and by Lemma 2.1, it follows that each module over a right noetherian ring has a $\mathcal{I}_n^{\perp}$-envelope, cf. [1].

4 Tilting and cotilting approximations

Tilting theory arised as a generalization of the classical Morita theory. Recall that two rings R and S are *Morita equivalent* provided that the categories Mod-R and Mod-S are equivalent:

$$\text{Mod-}R \quad \underset{G}{\overset{F}{\rightleftarrows}} \quad \text{Mod-}S.$$

So Morita equivalent rings have exactly the same properties defined by category theoretic properties of the full module categories. Morita equivalence is well-understood also ring theoretically: R is Morita equivalent to S iff there is $n < \omega$ and an idempotent matrix $e \in M_n(R)$ such that $S \cong eM_n(R)e$ and $M_n(R)eM_n(R) = M_n(R)$. In other words, $S \cong \text{End}(P_R)$ where P is a *progenerator* (= finitely generated projective generator for Mod-R). Moreover, $F \cong \text{Hom}_R(P, -)$ and $G \cong - \otimes_S P$, [2].

Tilting modules generalize the progenerators:

Definition 4.1 A module T is *tilting* provided that

1. $T \in \mathcal{P}_1$,
2. $\text{Ext}_R^1(T, T^{(\kappa)}) = 0$ for all cardinals κ, and
3. there is an exact sequence $0 \longrightarrow R \to T_1 \to T_2 \longrightarrow 0$ where $T_1, T_2 \in \text{Add}(T)$.

A ring S is called *tilted* from R if there is a tilting module T such that $S \cong \text{End}(T_R)$.

Of particular importance are the finitely generated tilting modules. The fundamental result of the tilting theory - the Tilting Theorem - [11], [33], [40], says that a finitely generated tilting module T induces a pair of category equivalences between $T^{\perp}$ and ${}^{\intercal}T$, and between $\text{Ker}\,\text{Hom}_R(T, -)$ and $\text{Ker}(- \otimes_S T)$, where $S = \text{End}(T_R)$:

$$T^{\perp} = \text{Ker}\,\text{Ext}_R^1(T, -) \quad \underset{-\otimes_S T}{\overset{\text{Hom}_R(T,-)}{\rightleftarrows}} \quad \text{Ker}\,\text{Tor}_1^S(-, T) = {}^{\intercal}T$$

$$\operatorname{Ker}\operatorname{Hom}_R(T,-) \quad \underset{\operatorname{Tor}_1^S(-,T)}{\overset{\operatorname{Ext}_R^1(T,-)}{\rightleftarrows}} \quad \operatorname{Ker}(- \otimes_S T)$$

In fact, the pairs $(T^{\perp}, \operatorname{Ker}\operatorname{Hom}_R(T,-))$ and $(\operatorname{Ker}(- \otimes_S T), {}^{\top}T)$ are torsion theories in Mod-R and Mod-S, respectively. So the pair of category equivalences is sometimes refered to as a *tilting counter-equivalence*, [13].

If S is tilted from R then Mod-S is not necessarily equivalent to Mod-R, but the properties of R and S are not that different: for example, if T is finitely generated then the global dimension of R and S differs at most by 1, R and S have isomorphic Grothendieck groups in case R and S are right artinian etc. We refer to [6] and [40] for further properties of finite dimensional tilting modules and tilted algebras.

Rather than investigating the tilting counter-equivalences we will aim at relations to the approximation theory. First, we need a characterization of tilting modules in terms of the classes they generate:

Lemma 4.2 *A module T is tilting iff* $\operatorname{Gen}(T) = T^{\perp}$. *In this case* $\operatorname{Pres}(T) = \operatorname{Gen}(T)$.

.

Proof. Assume T is tilting. Condition 4.1.2 says that $T^{(\kappa)} \in T^{\perp}$ for any cardinal κ, and 4.1.1 that $T^{\perp}$ is closed under homomorphic images. So $\operatorname{Gen}(T) \subseteq T^{\perp}$.

Let $M \in T^{\perp}$ and let $f : R^{(\lambda)} \to M$ be an epimorphism. Consider the exact sequence

$$0 \longrightarrow R^{(\lambda)} \xrightarrow{g} T_1^{(\lambda)} \to T_2^{(\lambda)} \longrightarrow 0$$

induced by condition 4.1.3. We form the pushout of f and g:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^{(\lambda)} & \xrightarrow{g} & T_1^{(\lambda)} & \longrightarrow & T_2^{(\lambda)} & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & T_2^{(\lambda)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $M \in T^{\perp}$, the second row splits, so M is a direct summand in G. Since h is surjective, $G \in \operatorname{Gen}(T)$, hence $M \in \operatorname{Gen}(T)$.

Conversely, assume that $\operatorname{Gen}(T) = T^{\perp}$. Let N be a module and E be its injective hull. Applying $\operatorname{Hom}_R(T,-)$ to $0 \longrightarrow N \to E \to E/N \longrightarrow 0$, we get $0 = \operatorname{Ext}_R^1(T, E/N) \to \operatorname{Ext}_R^2(T,N) \to \operatorname{Ext}_R^2(T,E) = 0$, since $T^{\perp}$ is closed under homomorphic images. So $\mathrm{Ext}^2_R(T,-)=0$, i.e., $T\in\mathcal{P}_1$. Condition 4.1.2 is clear by assumption.

It remains to prove that $\mathrm{Pres}(T)=\mathrm{Gen}(T)$. Let $M\in\mathrm{Gen}(T)$. Then the canonical map $\varphi\in\mathrm{Hom}_R(T^{(\mathrm{Hom}_R(T,M))},M)$ is surjective, so there is an exact sequence $0\longrightarrow K\to T^{(\mathrm{Hom}_R(T,M))}\xrightarrow{\varphi}M\longrightarrow 0$. Applying $\mathrm{Hom}_R(T,-)$, we get

$$0\to\mathrm{Hom}_R(T,K)\to\mathrm{Hom}_R(T,T^{(\mathrm{Hom}_R(T,M))})\overset{\mathrm{Hom}_R(T,\varphi)}{\to}\mathrm{Hom}_R(T,M)$$
$$\to\mathrm{Ext}^1_R(T,K)\to\mathrm{Ext}^1_R(T,T^{(\mathrm{Hom}_R(T,M))})=0.$$

By definition, $\mathrm{Hom}_R(T,\varphi)$ is surjective, so $\mathrm{Ext}^1_R(T,K)=0$ and $K\in\mathrm{Gen}(T)$, q.e.d. ∎

It follows that if T is tilting then $\mathrm{Gen}(T)$ is a torsion class of modules, called the *tilting torsion class* generated by T.

In the finite dimensional algebra case, finite dimensional tilting modules were characterized by Bongartz: they coincide with the modules of the form $\oplus_{i\leq r}M_i^{n_i}$ where $M_i\in\mathcal{P}_1$ is an indecomposable splitter, $M_i\not\cong M_j$ for all $i\neq j\leq r$, and r is the rank of the Grothendieck group of R, [10].

We will consider the Dedekind domain case:

Example 4.3 Let R be a Dedekind domain and P be a set of non-zero prime ideals of R. Put $R_{(P)}=\cap_{q\notin P}R_q$ where R_q denotes the localization of R at the prime ideal q. Put $T_P=\bigoplus_{p\in P}E(R/p)\oplus R_{(P)}$. Then T_P is a tilting module, [47].

Denote by $\mathcal{T}_P$ the class of all modules which are p-divisible for all $p\in P$. Then $\mathcal{T}_P=\mathrm{Gen}(T_P)$ is a tilting torsion class, cf. [45], [47].

Assuming V = L, if $R=\mathbb{Z}$ (or R is any small Dedekind domain) then the torsion classes of the form $\mathcal{T}_P$ are the only tilting torsion classes of modules over R, see [32] (or [47]). In particular, there is only a set of cotorsion theories of abelian groups cogenerated by tilting groups. This contrasts with the fact that (in ZFC) there is a proper class of complete cotorsion theories of abelian groups, [31].

Now, we will characterize tilting torsion classes among all torsion classes of modules in terms of approximations. We will work in a slightly more general setting of pretorsion classes: a class of modules, $\mathcal{C}$, is a *pretorsion class* provided that $\mathcal{C}$ is closed under arbitrary direct sums and homomorphic images.

Theorem 4.4 *Let $\mathcal{T}$ be a pretorsion class of modules. The following conditions are equivalent:*

1. *$\mathcal{T}$ is a tilting torsion class;*

2. *$\mathcal{T}$ is a special preenvelope class;*

3. *R has a special $\mathcal{T}$-preenvelope.*

Proof. 1. implies 2.: By Lemma 4.2, $\mathcal{T} = T^\perp$ for a tilting module T. So 2. follows by Theorem 2.2.2.

2. implies 3.: Trivial.

3. implies 1.: Let $0 \longrightarrow R \to T_1 \to T_2 \longrightarrow 0$ be a special $\mathcal{T}$-preenvelope of R. We will prove that $T = T_1 \oplus T_2$ is a tilting module such that $\mathrm{Gen}(T) = \mathcal{T}$.

Since $\mathcal{T}$ is a pretorsion class, we have $T \in \mathcal{T}$, and $\mathrm{Gen}(T) \subseteq \mathcal{T}$. Let $M \in T^\perp (= T_2^\perp)$. The pushout argument from the proof of Lemma 4.2 shows that $M \in \mathrm{Gen}(T)$. Finally, the $\mathcal{T}$-preenvelope of R is special, so $T_2 \in {}^\perp\mathcal{T}$, and $\mathcal{T} \subseteq ({}^\perp\mathcal{T})^\perp \subseteq T_2^\perp = T^\perp$.

This proves that $T^\perp = \mathrm{Gen}(T) = \mathcal{T}$, so T is tilting by Lemma 4.2. ∎

Corollary 4.5 *A pretorsion class $\mathcal{T}$ is a tilting torsion class iff $\mathcal{T} = T^\perp$ for a splitter T.*

Proof. By Theorems 2.2 and 4.4. ∎

If T is a tilting module then $({}^\perp\mathrm{Gen}(T), \mathrm{Gen}(T))$ is a complete cotorsion theory cogenerated by T. The structure of the class ${}^\perp\,\mathrm{Gen}(T)$ follows from Corollary 2.3: each element $M \in^\perp \mathrm{Gen}(T)$ is a direct summand in a module N such that N is an extension of a free module by a direct sum of copies of T. In fact, there is a more precise description available:

Theorem 4.6 *Let R be a ring and T be a tilting module. Let $0 \to R \to T_1 \to T_2 \to 0$ be a short exact sequence with $T_1, T_2 \in \mathrm{Add}(T)$ (cf. Definition 4.1). Denote by $\mathcal{X}_T$ the class of all direct summands of the modules X such that there exist cardinals κ, λ, and an exact sequence $0 \to R^{(\lambda)} \to X \to T_2^{(\kappa)} \to 0$.*

Then $\mathcal{X}_T = {}^\perp(T^\perp)$, so the cotorsion theory cogenerated by T is $\mathfrak{C}_T = (\mathcal{X}_T, \mathrm{Gen}(T))$. Moreover, $\mathcal{K}_{\mathfrak{C}_T} = \mathrm{Add}(T) \subseteq \mathcal{X}_T \subseteq \mathcal{P}_1 \cap {}^\perp\,\mathrm{Add}(T)$.

Proof. Let $M \in {}^\perp\,\mathrm{Gen}(T)$. Let $0 \to N \to F \to M \to 0$ be a short exact sequence with F free. Let κ be such that there exists an epimorphism $f : R^{(\kappa)} \to N$. Let ν be the embedding of $R^{(\kappa)}$ into $T_1^{(\kappa)}$. Consider the pushout of ν and f:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & R^{(\kappa)} & \xrightarrow{\nu} & T_1^{(\kappa)} & \longrightarrow & T_2^{(\kappa)} & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & T_2^{(\kappa)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

The second column gives $G \in \mathrm{Gen}(T_1) \subseteq \mathrm{Gen}(T)$. Next, consider the pushout of the monomorphisms $N \to F$ and $N \to G$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & G & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & T_2^{(\kappa)} & = & T_2^{(\kappa)} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

The second column gives $X \in \mathcal{X}_T$. The second row splits since $M \in {}^{\perp}\mathrm{Gen}(T)$. So $M \in \mathcal{X}_T$.

Conversely, assume that $M \in \mathcal{X}_T$, so M is a summand of some $X \in \mathcal{X}_T$ of the form $0 \to F \to X \to T_2^{(\kappa)} \to 0$ where F is free. Clearly, $F \in {}^{\perp}\mathrm{Gen}(T)$. Moreover, $T_2^{(\kappa)} \in {}^{\perp}\mathrm{Gen}(T)$, since $T \in {}^{\perp}\mathrm{Gen}(T)$. It follows that X, and hence M, belong to ${}^{\perp}\mathrm{Gen}(T)$.

Since $T \in \mathcal{P}_1$, all elements of $\mathcal{X}_T$ have projective dimension ≤ 1. Since $\mathrm{Add}(T) \subseteq \mathrm{Gen}(T)$, we have also $\mathcal{X}_T \subseteq {}^{\perp}\mathrm{Add}(T)$. On the other hand, $\mathrm{Add}(T) \subseteq \mathcal{X}_T$ as $T \in {}^{\perp}\mathrm{Gen}(T)$. It follows that $\mathrm{Add}(T) \subseteq \mathcal{K}_{\mathfrak{C}_T}$.

Finally, take $M \in \mathcal{K}_{\mathfrak{C}_T}$. By Lemma 4.2, there is an exact sequence $0 \to K \to T^{(\sigma)} \to M \to 0$, where σ is a cardinal and $K \in \mathrm{Gen}(T)$. Since $M \in {}^{\perp}\mathrm{Gen}(T)$, the sequence splits, so $M \in \mathrm{Add}(T)$. ∎

As in Corollary 2.3, we see that the complement G in the proof of Theorem 4.6 satisfies $G \in \mathrm{Add}(T)$.

Another interesting example of a tilting module is the Fuchs' divisible module [28], [29, VI.3]:

Example 4.7 Assume that R is a domain. Denote by δ the *Fuchs' divisible module*: $\delta \stackrel{\text{def}}{=} F/G$, where F is the free module with the free basis consisting of all k-tuples $(r_1, ..., r_k)$ where $k < \omega$ and $0 \neq r_i \in R$ $(i \leq k)$, and G is the submodule of F generated by all elements of the form $(r_1, ..., r_k)r_k - (r_1, ..., r_{k-1})$ $(k \geq 1)$. For $k = 0$, we have $w = (\emptyset) + G \in \delta$, $wR \cong R$, and δ/wR is torsion. Clearly, δ is a divisible module.

δ is a tilting module with the tilting torsion class $\mathrm{Gen}(\delta) = \mathcal{DI}$. Indeed, $\delta \in \mathcal{P}_1$ by [29, Lemma VI.3.1]. Also, for all κ, $\mathrm{Ext}_R(\delta, \delta^{(\kappa)}) = 0$ by [29, Proposition VI.3.4]. Consider the exact sequence $0 \to wR \to \delta \to \delta/wR \to 0$. Since $wR \cong R$ and δ/wR is isomorphic to a summand of δ by [29, p. 124], δ is tilting by Definition 4.1. By [29, Proposition VI.3.4], $\delta^\perp = \mathrm{Gen}(\delta)$ contains all divisible modules. On the other hand, δ, and hence every module in $\mathrm{Gen}(\delta)$, is divisible.

By Theorem 4.4, every module has a special $\mathrm{Gen}(\delta)$-preenvelope which is simply the special $\mathcal{DI}$-preenvelope of Theorem 3.4. Moreover, $\mathcal{K}_{\mathfrak{C}_\delta} = \mathrm{Add}(\delta)$ by Theorem 4.6.

By Theorem 3.7, $(\mathcal{P}_n, \mathcal{P}_n^\perp)$ is a complete cotorsion theory. It appears to be open to determine when the $\mathcal{P}_n^\perp$-preenvelopes have minimal versions (= envelopes) for $n \geq 1$. Our next theorem will give a negative answer for $n = 1$ in the particular case of Prüfer domains with $Q \notin \mathcal{P}_1$. First, we prove that $\mathcal{DI}$-envelopes may not exist in general (cf. Theorem 3.4.1):

Proposition 4.8 *Let R be a domain. Then the following are equivalent:*

1. *Each free module has a $\mathcal{DI}$-envelope.*
2. *R has a $\mathcal{DI}$-envelope.*
3. *$Q \in \mathcal{P}_1$.*

Proof. That 1. implies 2. is clear. Assume 2. Let $0 \to R \xrightarrow{\mu} D \to D/R \to 0$ be exact, where $\mu : R \hookrightarrow D$ is a (special) $\mathcal{DI}$-envelope of R. Take $0 \neq r \in R$. Since $Dr = D$, there exists $d \in D$ with $dr = 1$. Since μ is special, $\mathrm{Ext}_R(D/R, D) = 0$. It follows that there exists $\psi_r \in \mathrm{Hom}_R(D, D)$ such that $\psi_r(1) = d$. Denote by ϕ_r the endomorphism of D given by the multiplication by r. Then $\mu = \psi_r \phi_r \mu$. By assumption, $\psi_r \phi_r$ is an automorphism of D. This proves that ϕ_r is monic for all $0 \neq r \in R$, hence D is torsion free. By [29, Theorem VI.4.1], $D \cong Q^{(\kappa)}$ for a cardinal κ. By Lemma 1.3, we have

$\kappa = 1$. Since $Q/qR \cong Q/R$ for any $0 \neq q \in Q$, and μ is special, we infer from Theorem 4.6 that $Q/R \in \mathcal{P}_1$. The latter is equivalent to $Q \in \mathcal{P}_1$.

3. implies 1. Assume $Q \in \mathcal{P}_1$. Let $F = R^{(\kappa)}$ be a free module of rank κ and put $E = Q^{(\kappa)}$. We will prove that the inclusion $\nu : F \hookrightarrow E$ is a $\mathcal{DI}$-envelope of F. Since Q is $\sum$-injective, we have $\mathrm{Ext}^1_R(Q/R, Q^{(\lambda)}) = 0$ for any λ. Since $Q \in \mathcal{P}_1$, [29, Theorem VI.1.3] implies that $\mathcal{DI} = \mathrm{Gen}(Q)$, and clearly $Q/R \in \mathcal{P}_1$. So $\mathrm{Ext}^1_R(Q/R, D) = 0$ for any divisible module D, and ν is a special $\mathcal{DI}$-preenvelope of F.

Assume that φ is an endomorphism of E with $\varphi\nu = \nu$. Since F is essential in E, φ is monic. Since $\mathrm{Hom}_R(E, E) = \mathrm{Hom}_Q(E, E)$, $\varphi(E)$ is a Q-subspace of E containing F, hence φ is surjective. This proves that ν is a $\mathcal{DI}$-envelope of F. ∎

Similarly, FP-injective envelopes may not exists (cf. Theorem 3.4.2):

Theorem 4.9 *Assume R is a Prüfer domain.*

1. *The cotorsion theory cogenerated by δ is $\mathfrak{C}_\delta = (\mathcal{P}_1, \mathcal{DI})$, and $\mathcal{DI} = \mathcal{FI}$.*

2. *If* $\mathrm{proj.dim}(Q) \geq 2$, *then no free module has an $\mathcal{FI}$-envelope.*

Proof. 1. We prove that $\mathcal{P}_1 = {}^{\perp}\mathcal{DI}$. In view of Example 4.7 and Theorem 4.6, it suffices to prove that $\mathcal{P}_1 \subseteq {}^{\perp}\mathcal{DI}$: Each element of $\mathcal{P}_1$ is a union of a continuous chain of submodules such that all successive factors are finitely presented cyclic by [29, Corollary IV.4.7]. By Lemma 2.1, we get $\mathcal{P}_1^{\perp} = D^{\perp}$ where D is the direct sum of a representative set of all finitely presented cyclic modules. By [29, Proposition II.2.2], $D^{\perp} = \mathcal{DI}$. In particular, $\mathcal{P}_1 \subseteq {}^{\perp}\mathcal{DI}$.

Since any Prüfer domain is coherent, each finitely generated submodule of a finitely presented module is likewise finitely presented. So if F is finitely presented then there exist $n < \omega$ and a chain of submodules $0 = F_0 \subset F_1 \subset \cdots \subset F_n = F$ such that F_{i+1}/F_i is cyclic and finitely presented for all $i < n$. Since R is Prüfer, each finitely generated ideal is projective. By [29, Proposition II.2.2], we have $\mathcal{FI} = \{\oplus_{I \subseteq R, gen(I) < \omega} R/I\}^{\perp} = \mathcal{DI}$.

2. By part 1. and by Proposition 4.8. ∎

We turn to the dual case of cotilting modules. They generalize injective cogenerators:

Definition 4.10 A module C is *cotilting* provided that

1. $C \in \mathcal{I}_1$,

2. $\mathrm{Ext}^1_R(T^\kappa, T) = 0$ for all cardinals κ, and

3. there is an exact sequence $0 \longrightarrow C_1 \to C_2 \to W \longrightarrow 0$ where $C_1, C_2 \in \mathrm{Prod}(T)$ and W is an injective cogenerator for Mod-R.

If ${}_SU_R$ is a Morita bimodule (i.e., a faithfully balanced bimodule which is an injective cogenerator on either side) then U induces a Morita duality between Mod-R and S-Mod, [2]. Similarly, faithfully balanced cotilting bimodules induce a generalized Morita duality [15], [17].

Again, rather than investigating the dualities, we will aim at relations between cotilting and the approximation theory. First, we formulate the property of being a cotilting module in terms of classes of modules:

Lemma 4.11 *A module C is cotilting iff* $\mathrm{Cogen}(C) = {}^{\perp}C$. *In this case* *Copres*$(C) = \mathrm{Cogen}(C)$.

.

Proof. Dual to the proof of Lemma 4.2. ■

It follows that if C is cotilting then $\mathrm{Cogen}(C)$ is a torsion-free class of modules, called the *cotilting torsion-free class* cogenerated by C.

We would like to characterize cotilting torsion-free classes among all torsion-free classes of modules in terms of approximations. Again, we will work in a slightly more general setting of pretorsion-free classes: a class of modules, $\mathcal{C}$, is a *pretorsion-free class* provided that $\mathcal{C}$ is closed under submodules and arbitrary direct products.

The problem is that we do not have a dual of Theorem 2.2 (for cotorsion theories generated by a single module), or, more precisely, we do not know whether all cotilting modules are pure-injective (then we could simply use Theorem 2.8). But there is a way around: we make use of a generalized dual of the "Bongartz lemma", [44]:

Lemma 4.12 *Let R and S be rings. Let $A \in S$-Mod-R and $B \in$* Mod-R. *Denote by λ the number of generators of the left S-module* $\mathrm{Ext}^1_R(B, A)$. *Assume that* $\mathrm{Ext}^1_R(A^\lambda, A) = 0$. *Then there is a module $C \in$* Mod-R *such that*

1. $\mathrm{Ext}^1_R(C, A) = 0$ *and*

2. *there is an exact sequence* $0 \to A^\lambda \to C \to B \to 0$ *in* Mod-R.

Proof. We choose extensions $\mathcal{E}_\alpha = 0 \longrightarrow A \to E_\alpha \xrightarrow{\rho_\alpha} B \longrightarrow 0$ $(\alpha < \lambda)$ so that their equivalence classes generate $\mathrm{Ext}^1_R(B, A)$ as a left S-module. Let $0 \longrightarrow A^\lambda \xrightarrow{\mu} C \to B \longrightarrow 0$ be the extension obtained by a pullback of the direct product extension $0 \longrightarrow A^\lambda \to \prod_{\alpha<\lambda} E_\alpha \xrightarrow{\prod \rho_\alpha} B^\lambda \longrightarrow 0$ and of $\Delta_B \in Hom_R(B, B^\lambda)$ defined by $\Delta_B(b) = (b \mid \alpha < \lambda)$. For each $\alpha < \lambda$, we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A^{\lambda} & \longrightarrow & \prod_{\alpha<\lambda} E_\alpha & \xrightarrow{\prod \rho_\alpha} & B^{\lambda} & \longrightarrow & 0 \\
 & & \Vert & & {\scriptstyle\tau}\uparrow & & {\scriptstyle\Delta_B\uparrow\sigma_\alpha}\downarrow & & \\
0 & \longrightarrow & A^{\lambda} & \xrightarrow{\mu} & C & \longrightarrow & B & \longrightarrow & 0 \\
 & & {\scriptstyle\pi_\alpha}\downarrow & & {\scriptstyle h}\downarrow & & \Vert & & \\
0 & \longrightarrow & A & \xrightarrow{f} & X_\alpha & \xrightarrow{g} & B & \longrightarrow & 0 \\
 & & \Vert & & & & \Vert & & \\
0 & \longrightarrow & A & \longrightarrow & E_\alpha & \xrightarrow{\rho_\alpha} & B & \longrightarrow & 0
\end{array}
$$

where σ_α is the α-th projection of B^λ to B, and the third row is obtained by pushing out the second row along the α-th canonical projection, π_α, of A^λ onto A. Using the α-th projection, η_α, of $\prod_{\alpha<\lambda} E_\alpha$ onto E_α and the pushout property, we get $\varphi \in \mathrm{Hom}_R(X_\alpha, E_\alpha)$ making the lower left square commutative.

Since $\mathrm{Im}(f) = \mathrm{Ker}(g)$, $\mathrm{Im}(h) + \mathrm{Ker}(g) = X_\alpha$, and $gh = \sigma_\alpha(\prod \rho_\alpha)\tau = \rho_\alpha \eta_\alpha \tau = \rho_\alpha \varphi h$, we infer that also the lower right square is commutative. This means that the third and fourth rows are equivalent as extensions of A by B.

Consider the long exact sequence

$$
\begin{aligned}
&0 \to \mathrm{Hom}_R(B, A) \to \mathrm{Hom}_R(C, A) \to \mathrm{Hom}_R(A^\lambda, A) \xrightarrow{\delta} \\
&\mathrm{Ext}^1_R(B, A) \to \mathrm{Ext}^1_R(C, A) \xrightarrow{\mathrm{Ext}^1_R(\mu, A)} \mathrm{Ext}^1_R(A^\lambda, A) = 0
\end{aligned}
$$

induced by $\mathrm{Ext}^i_R(-, A)$. Since equivalence classes of the extensions $\mathcal{E}_\alpha$ ($\alpha < \lambda$) generate $\mathrm{Ext}^1_R(B, A)$, the commutative diagram constructed above shows that the connecting S-homomorphism δ is surjective. Hence, the S-homomorphism $\mathrm{Ext}^1_R(\mu, A)$ is a monomorphism. This proves that $\mathrm{Ext}^1_R(C, A) = 0$. ∎

Corollary 4.13 *Let C be a cotilting module. For each module M there are a cardinal λ, a module $C' \in \mathrm{Cogen}(C)$, and an exact sequence $0 \longrightarrow C^\lambda \to C' \to M \longrightarrow 0$.*

Proof. By Lemma 4.12 for $A = C$, $S = \mathbb{Z}$ and $B = M$. ∎

The exact sequence from Corollary 4.13 is called the *C-torsion-free resolution of M*, cf. [18].

Dual results to Lemma 4.12 and Corollary 4.13 hold true for arbitrary tilting modules T, cf. [19], [44] - in particular, each module has a *T-torsion resolution*. In fact, these are easy consequences of Theorem 2.2.

Now, we may characterize the cotilting torsion-free classes:

Theorem 4.14 *Let $\mathcal{F}$ be a pretorsion-free class of modules. Let W be an injective cogenerator for* Mod-R*. Then the following are equivalent:*

1. *$\mathcal{F}$ is a cotilting torsion-free class;*
2. *$\mathcal{F}$ is a special precover class;*
3. *W has a special $\mathcal{F}$-precover.*

Proof. 1. implies 2.: We have $\mathcal{F} = \text{Cogen}(C)$ for a cotilting module C. Let M be a module. Then the C-torsion-free resolution of M from Corollary 4.13 is a special $\mathcal{F}$-precover of M.

2. implies 3.: Trivial.

3. implies 1.: Let $0 \longrightarrow C_1 \to C_2 \to W \longrightarrow 0$ be a special $\mathcal{T}$-precover of W. A dual proof to that of Theorem 4.4 shows that $C = C_1 \oplus C_2$ is a cotilting module such that $\text{Cogen}(C) = \mathcal{C}$. ∎

Corollary 4.15 *A pretorsion-free class $\mathcal{F}$ is a cotilting torsion-free class iff $\mathcal{F} = {}^{\perp}C$ for a splitter C.*

Proof. The direct implication follows from Lemma 4.11. Conversely, since $\mathcal{F}$ is closed under products, we have $\text{Ext}^1_R(C^{\lambda}, C) = 0$, and Lemma 4.12 implies that each module has a special $\mathcal{F}$-precover. So $\mathcal{F}$ is cotilting by Theorem 4.14. ∎

By Theorem 4.14, if C is a tilting module then $(\text{Cogen}(C), \text{Cogen}(C)^{\perp})$ is a complete cotorsion theory generated by C. There is a description of the class $\text{Cogen}(C)^{\perp}$ dual to the one given in Theorem 4.6:

Theorem 4.16 *Let R be a ring and C be a cotilting module. Let $0 \to C_2 \to C_1 \to W \to 0$ be a short exact sequence with $C_1, C_2 \in \text{Prod}(C)$ and W an injective cogenerator for* Mod-R*. Denote by $\mathcal{Y}_C$ the class of all direct summands of the modules Y such that there exist cardinals κ, λ, and an exact sequence $0 \to C_2^{\lambda} \to Y \to W^{\kappa} \to 0$.*

Then $\mathcal{Y}_C = ({}^{\perp}C)^{\perp}$, so the cotorsion theory generated by C is $\mathfrak{G}_C = (\text{Cogen}(C), \mathcal{Y}_C)$. Moreover, $\mathcal{K}_{\mathfrak{G}_C} = \text{Prod}(C) \subseteq \mathcal{Y}_C \subseteq \mathcal{I}_1 \cap \text{Prod}(C)^{\perp}$.

Proof. Dual to the proof of Theorem 4.6, using Lemma 4.11 in place of Lemma 4.2. ∎

There is a way of constructing cotilting modules from the tilting ones. The idea comes from the finite dimensional case:

Example 4.17 Let R be a finite dimensional k-algebra over a field k. Let be T a finite dimensional tilting module. Then the k-vector space dual $T^* = \text{Hom}_k(T, k)$ is a cotilting left R-module. Indeed, the only non-trivial verification is for condition 4.10.2 for $\kappa \geq \omega$: one proves that $(T^*)^\kappa \in \mathrm{Add}(T^*)$ which follows for example from the fact that the endomorphism ring $S = \mathrm{End}({}_R T^*)$ is left coherent and right perfect and T^* is a finitely presented right S-module, cf. [37].

We have a similar result for arbitrary commutative rings and arbitrary dual modules:

Lemma 4.18 *Let R be a commutative ring and T be a tilting module. Let T^* be a dual module of T. Then T^* is cotilting iff $(T^{(\kappa)})^{**} \in \mathrm{Gen}(T)$ for each cardinal κ.*

Proof. Since $T \in \mathcal{P}_1$, we have $T^* \in \mathcal{I}_1$. From the short exact sequence $0 \to R \to T_1 \to T_2 \to 0$ with $T_1, T_2 \in \mathrm{Add}(T)$ we get $0 \to T_2^* \to T_1^* \to R^* \to 0$ with $T_1^*, T_2^* \in \mathrm{Prod}(T^*)$. Note that R^* is an injective cogenerator for Mod-R. It follows that T^* is cotilting iff $\mathrm{Ext}^1_R((T^*)^\kappa, T^*) = 0$ for all cardinals κ. Applying [12, VI.5.1], we have

$$\begin{aligned}\mathrm{Ext}^1_R((T^*)^\kappa, T^*) = 0 \text{ iff } \mathrm{Tor}_1^R((T^*)^\kappa, T) = 0 \text{ iff } \mathrm{Tor}_1^R(T, (T^*)^\kappa) = 0 \\ \text{iff } \mathrm{Tor}_1^R(T, (T^{(\kappa)})^*) = 0 \text{ iff } \mathrm{Ext}^1_R(T, (T^{(\kappa)})^{**}) = 0.\end{aligned}$$

■

We will finish by an explicit construction of cotilting modules over commutative rings following [45]. First, we recall some relations between modules and their duals:

Lemma 4.19 *Let R be an S-algebra and M be a module.*

1. *M is flat iff M^* is injective;*
2. *Let I be a right ideal. Let M be a left R-module. Then M is I-torsion-free iff M^* is I-divisible. In particular, $M \in \mathcal{TF}$ iff $M^* \in \mathcal{DI}$.*
3. *Let I be a right ideal of R such that I has a projective resolution consisting of finitely generated projective modules. Then M is I-divisible iff M^* is I-torsion-free. If R is a domain then $M \in \mathcal{DI}$ iff $M^* \in \mathcal{TF}$.*
4. *Assume R is right coherent. Then M is I-divisible for all finitely generated ideals I iff M^* is flat.*

Proof. 1. and 2. are well-known (cf. [12, VI.5.1]).
3. and 4. M is I-divisible iff $\mathrm{Ext}^1_R(R/I, M)) = 0$ iff $\mathrm{Tor}_1^R(R/I, M^*) = 0$, by [12, Remark VI.5.3]. 4. now follows from the Flat Test Lemma [2]. ■

Theorem 4.20 *Let R be a commutative ring. Let $\mathcal{I}$ be a set of finitely generated projective ideals of R. Denote by $\mathcal{T}_I$ the class of all modules which are I-divisible for all $I \in \mathcal{I}$, and by $\mathcal{F}_I$ the class of all modules which are I-torsion-free for all $I \in \mathcal{I}$. Then $M^* \in \mathcal{T}_I$ iff $M \in \mathcal{F}_I$, for each module M.*

Moreover, $\mathcal{T}_I$ is a tilting torsion class and $\mathcal{F}_I$ is a cotilting torsion-free class closed under direct limits. Denote by T a tilting module generating $\mathcal{T}_I$.

Then each module has a special $\mathcal{T}_I$-preenvelope, a $\mathcal{Y}_{T^*}$-envelope, a special $\mathcal{X}_T$-precover, and an $\mathcal{F}_I$-cover.

Proof. Put $N = \oplus_{I \in \mathcal{I}} R/I$. Then $\mathcal{T}_I = N^\perp$, so each module has a special $\mathcal{T}_I$-preenvelope. Since N has projective dimension ≤ 1, $N^\perp$ is closed under homomorphic images. Since each $I \in \mathcal{I}$ is finitely generated, $N^\perp$ is closed under direct sums. By Theorem 4.4, $N^\perp$ is a tilting torsion class.

Let T be a tilting module with $\mathrm{Gen}(T) = \mathcal{T}_I$. By Lemma 4.18 and by parts 2. and 3. of Lemma 4.19, T^* is a cotilting module. By Theorem 2.8, every module has a $\mathrm{Cogen}(T^*)$-cover. Finally, for each module M, $\mathrm{Ext}^1_R(T, M^*) = 0$ iff $\mathrm{Tor}^R_1(T, M) = 0$ iff $\mathrm{Tor}^R_1(M, T) = 0$ iff $\mathrm{Ext}^1_R(M, T^*) = 0$ by [12, VI.5.1]. It follows that $M^* \in \mathcal{T}_I$ iff $M \in \mathrm{Cogen}(T^*)$. So $\mathrm{Cogen}(T^*) = \mathcal{F}_I$ by Lemma 4.19.2. ■

Corollary 4.21 *Let R be a domain. Let δ be the Fuchs' divisible module. Let $\mathcal{I}$ be the set of all finitely generated projective ideals of R. Then δ^* is a cotilting torsion-free module, $\mathcal{T}_I = \mathrm{Gen}(\delta) = \mathcal{DI}$, $\mathcal{F}_I = \mathrm{Cogen}(\delta^*) = \mathcal{TF}$ and $\mathcal{Y}_{\delta^*} = \mathcal{WC}$. The kernel $\mathcal{K}_{\mathfrak{C}_{\delta^*}}$ is the class of all torsion-free pure-injective modules of injective dimension ≤ 1. If R is a Prüfer domain then $\mathcal{X}_\delta = \mathcal{P}_1$.*

Proof. By [29, II.2.2], $\mathcal{T}_I$ is the class of all divisible modules. The rest follows by Theorem 4.20 and Examples 3.2 and 4.7. ■

There is an immediate corollary for Dedekind domains (cf. Example 4.3):

Corollary 4.22 *Let R be a Dedekind domain. Let P be a set of maximal ideals of R. Denote by $\mathcal{T}_P$ the class of all modules which are I-divisible for all $I \in P$, and by $\mathcal{F}_P$ the class of all modules which are I-torsion-free for all $I \in P$. Then $\mathcal{T}_P$ is a tilting torsion class, and $\mathcal{F}_P$ is a cotilting torsion-free class. Every module has a special $\mathcal{T}_P$-preenvelope and an $\mathcal{F}_P$-cover.*

In fact, if R is a Dedeekind domain then the torsion-free classes of the form $\mathcal{F}_P$ for a set of maximal ideals P are the only cotilting torsion-free classes closed under arbitrary direct limits. Moreover, $\mathcal{F}_P = \mathrm{Cogen}(C_P)$, where $C_P = \bigoplus_{p \notin P} E(R/P) \oplus \prod_{p \in P} \hat{R}_p$, where $\hat{R}_p$ denotes the completion of the localization of R at p, cf. [23].

There are many other aspects of the tilting and cotilting theory not even mentioned in theses notes: the endo-structure of the tilting/cotilting modules, their various finite and partial versions etc. For example, [3] contains a recent treatment of these aspects.

5 Open problems

1. Characterize the modules M such that ${}^{\perp}M$ is a special precover class.

By Theorem 2.8, these include all pure-injective modules. Assuming V = L, these include all modules over any right hereditary ring, cf. Example 2.4. In particular: *Is ${}^{\perp}\mathbb{Z}$ (= the class of all Whitehead groups) a special precover class (in ZFC)?*

2. Are all cotilting modules pure-injective?

This is of course true of the cotilting modules produced by a duality as in Lemma 4.18, and in fact of all known examples of cotilting modules. Angeleri-Hügel, Mantese and Tonolo proved that a cotilting module C is pure-injective iff $\mathrm{Cogen}(C)$ is closed under arbitrary direct limits.

3. Let R be a domain. When is ${}^{\perp}\mathcal{MC}$ a cover class?

This is true when R is a Dedekind domain, since then ${}^{\perp}\mathcal{MC} = \mathcal{FL}$. Otherwise, $\mathcal{P}_0 \subsetneq {}^{\perp}\mathcal{MC} \subsetneq \mathcal{FL}$, so ${}^{\perp}\mathcal{MC}$ is not closed under arbitrary direct limits. Note that $({}^{\perp}\mathcal{MC}, \mathcal{MC})$ is just the complete cotorsion theory cogenerated by Q, so ${}^{\perp}\mathcal{MC}$ is always a special precover class by Theorem 2.2. Moreover, $\mathcal{MC}$ is an envelope class by Example 3.2. The elements of ${}^{\perp}\mathcal{MC}$ are called *strongly flat*, [45].

4. Let T be a tilting module. When is $\mathrm{Gen}(T)$ *an envelope class?*

This is true if $\mathrm{Gen}(T) = \mathcal{C}^{\perp}$ where $\mathcal{C}$ is closed under extensions and arbitrary direct limits and $T \in \mathcal{C}$, by the existence of T-torsion resolutions and by Theorem 1.14. Proposition 4.8 shows that $\mathrm{Gen}(T)$ may not be an envelope class in general.

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